[redacted]

Class 12 maths [Application of Derivatives]

Rate of change (परिवर्तन की दर)

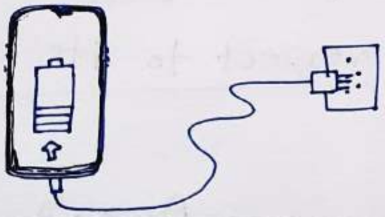
$y = f(x)$

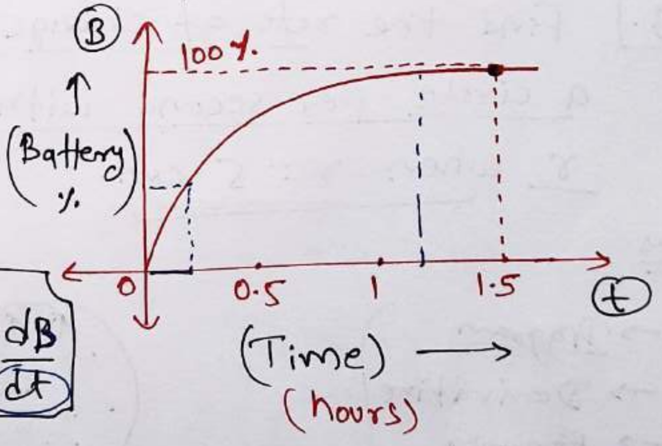
$\frac{dy}{dx} = f'(x) =$ rate of change of y with respect to x

✲ $\left.\frac{dy}{dx}\right|_{x=x_0} = f'(x_0) =$ rate of change of y with respect to x at $x = x_0$.

[Instantaneous (at some instant) / तात्क्षणिक (किसी क्षण पर) (किसी पल में)]

e.g. Charging Speed of a Mobile Phone.





$$\boxed{\text{Charging speed} = \text{Rate of change in Battery \% w.r.t. time} = \frac{dB}{dt}}$$

Charging speed in starting > Charging speed in the end

$$\left.\frac{dB}{dt}\right|_{t=0.5} > \left.\frac{dB}{dt}\right|_{t=1}$$

Questions on Rate of Change

How to Deal?

order

D → Diagram

D → Derivatives (Given / asked)

F → Formula

D → Differentiation

C → Constants.

Note:

$\boxed{\frac{dy}{dx} = \oplus ve} \Rightarrow$

ⓧ ↑	ⓨ ↑

$\boxed{\frac{dy}{dx} = \ominus ve} \Rightarrow$

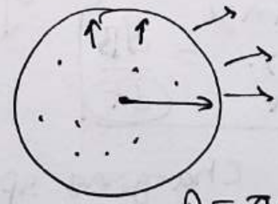
ⓧ ↑	ⓨ ↓

e.g. Find the rate of change of the area of a circle per second with respect to its radius r when $r = 5$ cm.

Ans.

D → Diagram
D → Derivative
F → Formula
D → Diff.
C → Constant.

Area = A



$\frac{dA}{dr} = ?$

$A = \pi r^2$

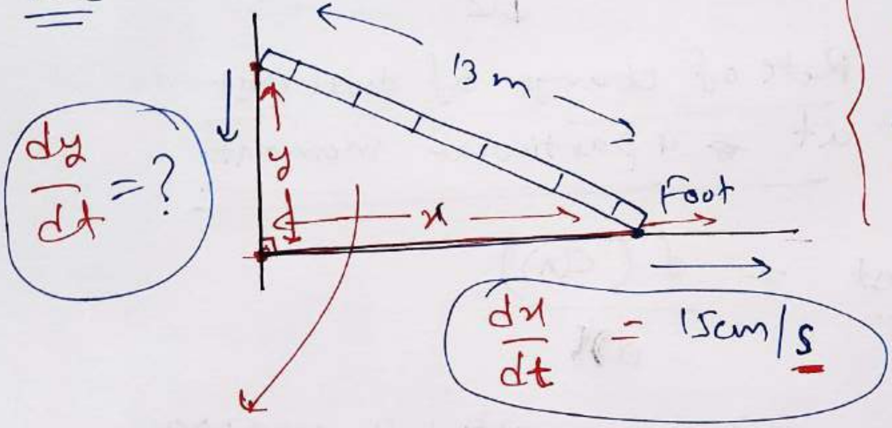
by Differentiating w.r.t. r

$$\frac{dA}{dr} = \pi(2r) \qquad (r = 5\text{cm})$$

$$\left.\frac{dA}{dr}\right|_{r=5} = 10\pi$$

e.g. A ladder 13 m long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 15 cm/s. How fast is its height on the wall decreasing when the foot of the ladder is 12 m away from the wall?

Ans.



D → Diagram ✓
D → Derivatives ✓
F → Formula. ✓
D → Diffⁿ.
C → Constants

Pythagoras Theorem:

$$x^2+y^2=13^2$$

by Diff. w.r.t. 't'

$$\Rightarrow \cancel{2}x\cdot\frac{dx}{dt}+\cancel{2}y\cdot\boxed{\frac{dy}{dt}}=0$$

$$\Rightarrow \frac{dy}{dt} = -\left(\frac{\boxed{x}\cdot\frac{dx}{dt}}{y}\right) = -\frac{(12\cancel{00}\times\overset{3}{\cancel{15}})}{(\cancel{500})} = -36 \text{ cm/s}$$

$x = 12$ m

$x^2+y^2=13^2$

$\Rightarrow 12^2+y^2=13^2$

$\boxed{y=5 \text{ m}}$

e.g. The total cost $C(x)$ in rupees, associated with the production of x units of an item is given by $\boxed{C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000}$.

Find the ~~Marginal~~ marginal cost when 3 units are produced.

(सीमांत लागत)

Ans.

Rate of change of total cost at a particular moment.

$$\text{Marginal cost} = \frac{d(C(x))}{dx}$$

$$C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$$

$$\frac{d(C(x))}{dx} = (0.005)(3x^2) - 0.02(2x) + (30)(1) + 0$$

$(x = 3)$

$$\frac{d(C(3))}{dx} = (0.005)(27) - (0.02)(6) + 30$$

$$= 0.135 - 0.12 + 30$$

$$= \underline{30.015} \text{ ₹}$$

$x \to$ units

Class 12 Maths [Application of Derivatives]

Exercise 6.1 Chapter ⑥ (Rate of change)

(order Fixed) ↓

D → Diagram
D → Derivatives (Given / Asked)
F → Formula
D → Differentiation
C → Constant

Q.1 Rate of change of area of a circle with respect to radius r

(i) when r = 3 cm (Constants) (ii) r = 4 cm.

Ans.

r $\quad \frac{dA}{dr} = ?$

$A = \pi r^2$

diff. w.r.t. 'r'

$$\frac{dA}{dr} = \pi(2r)$$

(i) $r = 3$

$$\frac{dA}{dr} = \pi(2 \times 3)$$
$$= \underline{6\pi} \text{ cm}^2/\text{cm}$$

(ii) $r = 4$ cm

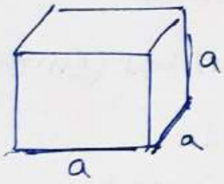
$$\frac{dA}{dr} = \pi(2 \times 4)$$
$$= 8\pi \text{ cm}^2/\text{cm}$$

Q.2 Volume of a cube is increasing at 8 cm³/s.

rate of change of area = ? , when edge = 12 cm.

(a = 12 cm)

Ans.



$\frac{dv}{dt} = 8$ cm³/sec.

Cube Area = S , $\frac{dS}{dt} = ?$

Volume of a cube

$v = a^3$

diff. w.r.t 't'

$\frac{dv}{dt} = 3a^2 \cdot \frac{da}{dt}$

$8 = 3(12)^2 \cdot \boxed{\frac{da}{dt}}$

$\Rightarrow \frac{8}{3(12)^2} = \frac{da}{dt}$

(a=12)

Area of a Cube (S)

$S = 6a^2$

diff. w.r.t. 't'

$\frac{dS}{dt} = 12a \cdot \boxed{\frac{da}{dt}}$

$\frac{dS}{dt} = \cancel{12(12)} \cdot \frac{8}{3\cancel{(12)^2}}$

$\frac{dS}{dt} = \frac{8}{3}$ cm²/sec.

Q.3 Radius increasing at 3 cm/s. $= \frac{dr}{dt}$

Rate of area change = ? $= \frac{dA}{dt}$ (when r = 10 cm)

$A = \pi r^2$

by Diff. w.r.to 't'

$\frac{dA}{dt} = \pi(2r) \cdot \frac{dr}{dt}$

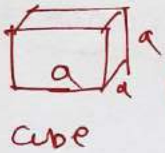
$= \pi(2\times10) \cdot (3) = 60\pi$ cm²/sec.

Q.4 Edge of a variable cube is increasing at 3 cm/s.

Rate of change of volume = ? = $\frac{dv}{dt}$ $\qquad \frac{da}{dt}$

(when edge = 10 cm)

$\boxed{a = 10}$



(edge = a) $\quad V = a^3$

by diff. w.r.to 't'

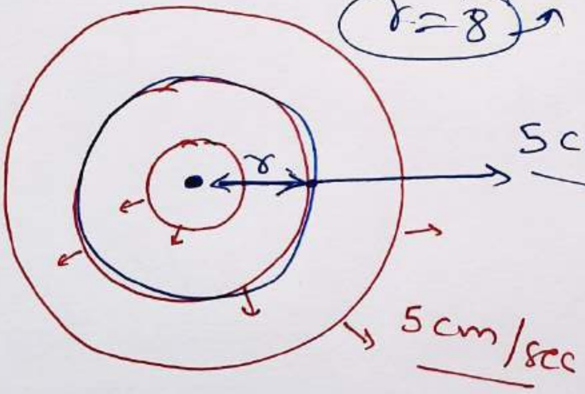
$$\frac{dv}{dt} = 3a^2\left(\frac{da}{dt}\right) = 3(10)^2.(3) = 900 \text{ cm}^3/\text{sec.}$$

Q.5 Waves moves in circles at the speed of 5cm/s.

~~note~~ How fast the enclosed area increasing?

(when radius = 8 cm) $\quad \boxed{r = 8}$ $\qquad \frac{dA}{dt}$ $\qquad \frac{dr}{dt}$

Ans.



$5\text{ cm/s} = \frac{dr}{dt}$

~~Area~~ Area = $A = \pi r^2$

$$\boxed{\frac{dA}{dt}} = \pi(2r).\frac{dr}{dt}$$

$$= \pi(2 \times 8).(5)$$

$$= 80\pi \text{ cm}^2/\text{sec.}$$

$\boxed{Q.6}$ radius of a circle increasing at the rate of $\boxed{0.7\ cm/s.} = \frac{dr}{dt}$

Rate of change of its Circumference = ?

C

$\frac{dc}{dt}$

$C = 2\pi r$

by diff. w.r.t. 't'

$\frac{dc}{dt} = 2\pi \frac{dr}{dt} = 2\pi(0.7)$

$= \underline{1.4\pi\ cm/sec.}$

[redacted]

Class 12 maths [Application of Derivatives]

Exercise 6.1 Chapter - 6 (ROC)

D → Diagram
D → Derivatives (Given/Asked)
F → Formula
D → Differentiation
C → Constants

Order

Q.7

Length (x) decreasing

Rate = 5 cm/min.

$$\boxed{\frac{dx}{dt} = -5}$$

width (y) increasing

Rate = 4 cm/min.

$$\boxed{\frac{dy}{dt} = 4}$$

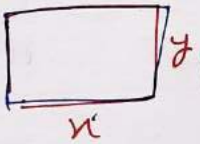
when $x = 8$ cm, $y = 6$ cm

Constants.

(i) Rate of change of Perimeter

Perimeter = P

$$\frac{dP}{dt} = ?$$



$\left(\frac{dy}{dt} = 4\right)$

$\left(\frac{dx}{dt} = -5\right)$

$$P = 2(x+y)$$

by diff. w.r.t. 't'

$$\frac{dP}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right)$$

$$= 2(-5+4)$$

$$= -2 \text{ cm/min.}$$

(ii) Rate of change of Area → A

$$\frac{dA}{dt} = ?$$

$$A = x.y$$

by diff. w.r.t. 't'

$$\frac{dA}{dt} = y.\frac{dx}{dt} + x.\frac{dy}{dt}$$

$$\frac{dA}{dt} = 6(-5) + 8(4)$$

$$= -30 + 32$$

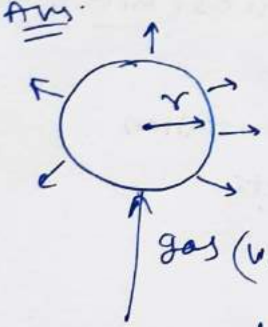
$$= 2 \text{ cm}^2/\text{min.}$$

$\boxed{Q.8}$ Spherical Balloon, Inflated by pumping 900 cubic cm of gas per second, Rate of change of radius = ?

when $r = 15$ cm.

$\left(\frac{dr}{dt}\right)$

Ans.



$\frac{dr}{dt} = ?$

gas (volume)

$\downarrow$

$\frac{dv}{dt} = 900\ cm^3/sec.$

(Spherical)

volume $= V = \frac{4}{3}\pi r^3$

by diff. w.r.to 't'

$\frac{dv}{dt} = \frac{4}{\cancel{3}}\pi(\cancel{3}r^2) \cdot \frac{dr}{dt}$

$\Rightarrow 900 = 4\pi(15)^2 \cdot \boxed{\frac{dr}{dt}}$

$\Rightarrow \frac{\cancel{900}\,1}{\cancel{900}\pi} = \frac{dr}{dt} = \frac{1}{\pi}\ cm/sec.$

$\boxed{Q.9}$ Spherical Balloon, Rate of change of Volume with radius = ?

when the later is 10 cm.

Ans. radius $= r = 10$ cm.

$\frac{dv}{dr} = ?$

volume of a sphere

$V = \frac{4}{3}\pi r^3$

by diff. w.r.t. 'r'

$\Rightarrow \frac{dV}{dr} = \frac{4}{\cancel{3}}\pi(\cancel{3}r^2)$

$(r = 10\ cm)$

$\frac{dv}{dr} = 4\pi(r^2) = 400\pi\ cm^3/cm.$

Q.10 Ladder 5 m long., leaning against a wall.

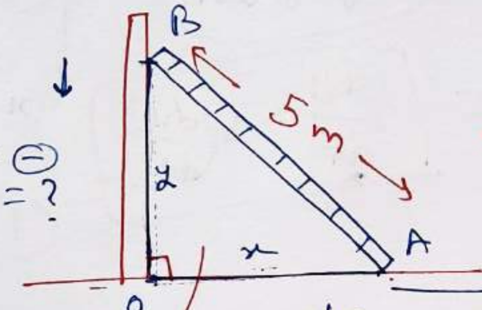
Bottom is pulled away from the wall at 2 cm/s.

How fast is its height decreasing = ?

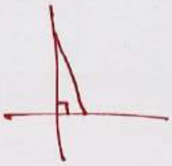
when Foot of ladder is 4 m aways from wall.

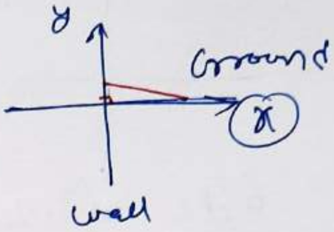
Ans.

$\frac{dy}{dt} = ⊖ = ?$ ↓



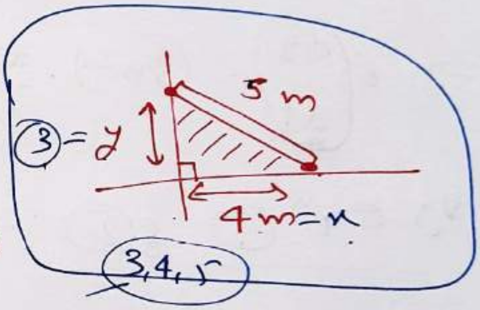
$\frac{dx}{dt} = 2cm/s$





Right angled Triangle,

By ~~Pt~~ Pythagoras Theorem!



3,4,5

$x = 4m = 400$ cm

$y = 3m = 300$ cm

$\frac{dx}{dt} = 2$ cm/s

$$x^2+y^2=5^2$$

by diff. w.r.to 't'

$$\cancel{2}x\frac{dx}{dt} + \cancel{2}y\cdot\boxed{\frac{dy}{dt}} = 0$$

$$\Rightarrow y\cdot\boxed{\frac{dy}{dt}} = -x\cdot\frac{dx}{dt}$$

$$\Rightarrow \boxed{\frac{dy}{dt}} = -\frac{x}{y}\cdot\frac{dx}{dt} = -\frac{4\cancel{00}}{3\cancel{00}}\times 2 = \boxed{\frac{-8}{3}\text{ cm/s}}$$

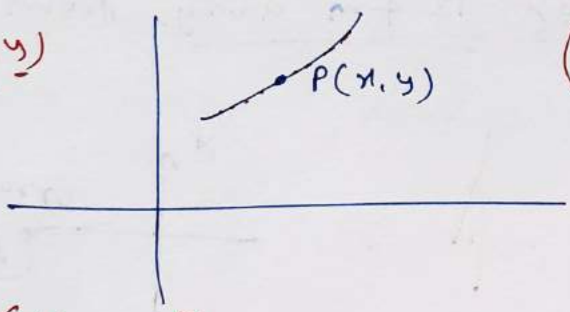
Decreases at $\boxed{\frac{8}{3}\text{ cm/s}}$ ✓

Q.11 Particle moves along $\boxed{6y = x^3+2}$.

Find points on the curve at which y-coordinate is changing 8 times as fast as the x-coordinate.

Ans.

(x, y)



Rate of change of 'y-coor' = 8 (Rate of change of x-coor)

$$\boxed{\frac{dy}{dt} = 8\left(\frac{dx}{dt}\right)} \text{ — } ①$$

$6y = x^3+2$

by diff. w.r.to 't'

$$\Rightarrow 6\left(\frac{dy}{dt}\right) = (3x^2)\cdot\frac{dx}{dt} \text{ — } ②$$

by eqn ① & ② :→ $\underset{2}{\cancel{6}}\left(8\cancel{\frac{dx}{dt}}\right) = \cancel{3}x^2\cdot\cancel{\frac{dx}{dt}}$

$\Rightarrow 16 = x^2$

$\boxed{x = \pm 4}$

$\boxed{x = 4}$

$6y = x^3+2$

$\Rightarrow 6y = 4^3+2$

$\Rightarrow \cancel{6}y = \underset{11}{\cancel{66}}$

$\boxed{y = 11}$

(4, 11)

$\boxed{x = -4}$

$6y = (x)^3+2$

$\Rightarrow 6y = -64+2$

$\Rightarrow \underset{3}{\cancel{6}}y = \underset{31}{\cancel{-62}}$

$\boxed{y = -31/3}$

$\left(4, -\frac{31}{3}\right)$

[redacted] Class 12 Maths | Application of Derivatives |

Exercise | 6.1 | Chapter | 6 |

(Rate of change)

Q.12 Radius of an air bubble is increasing at the rate of $\frac{1}{2}$ cm/s. At what rate is the volume of the bubble increasing when radius is 1 cm?

Ans.

$\frac{dr}{dt} = \frac{1}{2}$ cm/s $\qquad \frac{dv}{dt} = ?$

$V = \frac{4}{3}\pi r^3$

by differentiating w.r.to 't'

$$\frac{dv}{dt} = \frac{4}{3}\pi(3r^2)\cdot\frac{dr}{dt}$$

$r = 1 \qquad \frac{dr}{dt} \rightarrow \frac{1}{2}$

$$\Rightarrow \frac{dv}{dt} = \frac{\cancel{4}^{2}}{\cancel{3}}\cdot\pi\cdot\cancel{3}\times 1\cdot\frac{1}{\cancel{2}} = 2\pi \text{ cm}^3/\text{sec.}$$

Q.13 A balloon (always spherical), has a variable diameter $\frac{3}{2}(2x+1)$, Find rate of change of its volume with respect to x.

Ans. Dia. $= \frac{3}{2}(2x+1)$

$r = \frac{3}{4}(2x+1)$

$V = \frac{4}{3}\pi r^3$

$V = \frac{4}{3}\pi\left(\frac{3}{4}\cdot(2x+1)\right)^3$

$$V = \frac{4}{3}\pi\left(\frac{3}{4}\right)^3\cdot(2x+1)^3 = \left(\frac{3}{4}\right)^2\pi\cdot(2x+1)^3 \rightarrow$$

$$V = \left(\frac{3}{4}\right)^2 \cdot \pi \cdot (2x+1)^3 \qquad \frac{d(x^3)}{dx} = 3x^2$$

by diff. w.r.t. 'x'

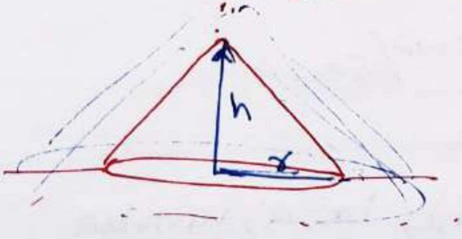
$$\frac{dv}{dx} = \frac{9}{\cancel{16}_{8}} \cdot \pi \cdot 3(2x+1)^2 \cdot \cancel{(2+0)}$$

$$= \frac{27}{8}\pi(2x+1)^2$$

Q.14 Sand is pouring at the rate of 12 cm³/s. The falling sand forms a cone on the ground in such a ways that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm?

(Constant)

$\frac{dv}{dt} = 12\ cm^3/s.$



$h = \frac{1}{6}$ of r

$h = \frac{r}{6}$ (Given)

$6h = r$

$\frac{dh}{dt} = ?$

Volume of cone $= V = \frac{1}{3}\pi r^2 h$

$\Rightarrow V = \frac{1}{3}\pi(6h)^2 \cdot h$

$\Rightarrow \boxed{V = 12\pi h^3}$

by diff. w.r.to 't'

$$\Rightarrow \underset{\underset{12}{\uparrow}}{\frac{dv}{dt}} = 12\pi(3\underset{\underset{h=4}{\uparrow}}{h^2}) \cdot \frac{dh}{dt}$$

$$\overset{1}{\cancel{12}} = \cancel{12}\pi(3 \cdot 4^2) \cdot \frac{dh}{dt}$$

$$\Rightarrow \frac{1}{48\pi} = \frac{dh}{dt}$$

cm/sec.

Q.15 Total cost C(x) in Rupees

$$\boxed{C(x) = 0.007x^3 - 0.003x^2 + 15x + 4000}$$

Marginal cost = ? when 17 units are produced.

↘ Rate of change of total cost

Ans.

$$\text{marginal cost} = \frac{d(\text{total cost})}{dx} = \frac{d(C(x))}{dx}$$

$$= \frac{d(0.007x^3 - 0.003x^2 + 15x + 4000)}{dx}$$

$$= 0.007.(3x^2) - 0.003(2x) + 15$$

(when $x=17$)

$$= 0.007.(3 \times 289) - 0.003(34) + 15$$

$$= 6.069 - 0.102 + 15$$

$$= 21.069 - 0.102 = \underline{20.967}$$

Q.16 Total Revenue R(x) sale x units

Find marginal revenue when $x = 7$

Ans. $\boxed{R(x) = 13x^2 + 26x + 15.}$

$$\text{Marginal Revenue} = \frac{d(R(x))}{dx}$$

$$= \frac{d(13x^2 + 26x + 15)}{dx}$$

$$= 13(2x) + 26$$

$x = 7$ put

$$MR = 13 \times (2 \times 7) + 26$$

$$= 13 \times 14 + 26$$

$$= 182 + 26$$

$$= \underline{208} ₹$$

Q. 17 Rate of change of the area of a circle with respect its radius r at $r = 6$ cm is —

(A) 10π [✓ (B) 12π] (C) 8π (D) 11π.

Ans. Area of a circle $= A = \pi r^2$

$$\Rightarrow \frac{dA}{dr} = \pi(2r)$$

$(r = 6 \text{ cm.})$

$$\frac{dA}{dr} = \pi(2 \times 6) = 12\pi$$

Q.18 The total revenue in rupees ~~received~~ received from the sale of x units of a product is given by $\boxed{R(x) = 3x^2 + 36x + 5}$. The marginal revenue, when $x = 15$ is —

(A) 116 (B) 96 (C) 90 [✓ (D) 126]

$$\text{marginal Revenue} = \frac{d\,(\overset{R(x)}{\text{total Revenue}})}{dx}$$

$$= \frac{d\,(3x^2 + 36x + 5)}{dx} = 3(2x) + 36$$

when $x = 15$

$$\text{M.R.} = 6(15) + 36 = 90 + 36 = 126 \text{ ₹}$$

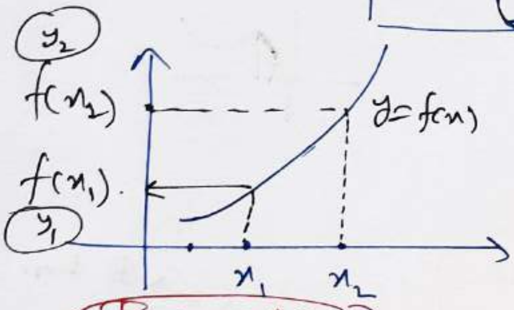
[redacted]

Increasing and Decreasing Functions.

(वर्धमान) (ह्रासमान)

Always move from left to Right

$x \uparrow$



Increasing

$x_1 < x_2$

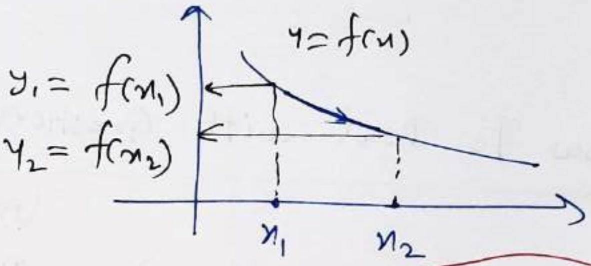
$$f(x_1) < f(x_2)$$

$$\frac{y_2 - y_1}{x_2 - x_1} = \text{Slope}$$

$$= \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{+}{+} = +$$

Slope = +

$$\boxed{f'(x) > 0}$$



$x_1 < x_2$ Decreasing

$$f(x_1) > f(x_2)$$

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad \frac{-}{+}$$

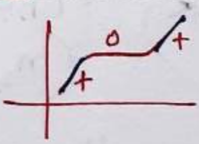
$$= -$$

Slope = −

$$\boxed{f'(x) < 0}$$

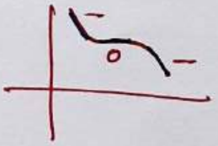
Increasing

$f'(x) \geq 0$



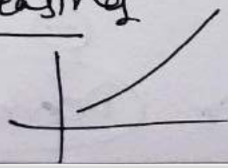
Decreasing

$f'(x) \leq 0$



Strictly Increasing

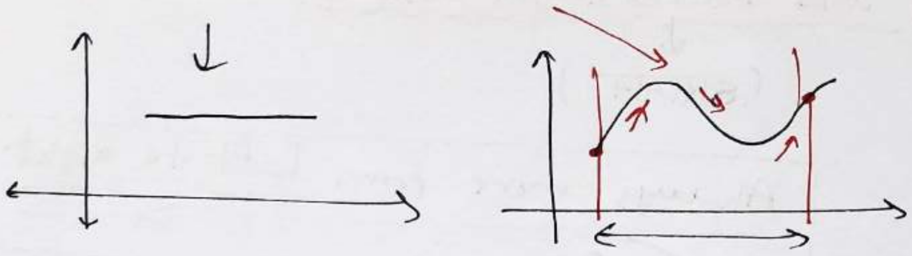
$f'(x) > 0$

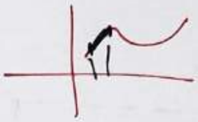


Strictly Decreasing

$f'(x) < 0$

Neither Increasing Nor Decreasing.





How To Deal with Questions?

Type-I → Interval (Given), ↑/↓ (Asked) → $f'(x)$ → ⊕ In. / → ⊖ ↓

$f'(x)$ ↓ (sign check)

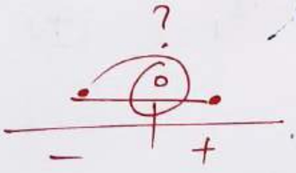
Type-II → Interval (Asked), ↑/↓ (Given) → $f'(x) = 0$

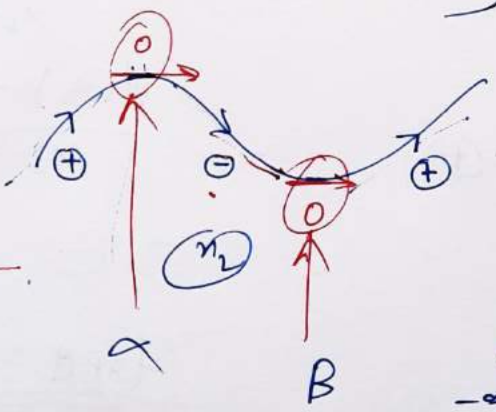
↓

Roots $x = ?$

↓ (α, β)

Numberline





Let Check (x_1)	Check (x_2)	Check (x_3)
$f'(x_1)$ ⊕	$f'(x_2)$ ⊖	$f'(x_3)$ ⊕
Increasing	Decreasing	⊕

e.g. Show that $f(x) = 3x^3 + 7$ is increasing on R.

Ans.

$f(x) = 3x^3 + 7$

$f'(x) = \boxed{9x^2}$ $\quad x \in R$ $\quad \boxed{(\ )^2 \geq 0}$

$x^2 \geq 0$

$\boxed{9x^2} \geq 0$

$f'(x) \geq 0 \Rightarrow \therefore f(x)$ is always increasing on R.

$\boxed{\text{Slope} = \boxed{+}}$

e.g. Find the interval in which $f(x) = 4x^3 - 6x^2 - 72x + 30$ is Ⅰ Increasing → $f'(x) > 0$

Ⅱ Decreasing → $f'(x) < 0$

Ans. $f'(x) = 12x^2 - 12x - 72$

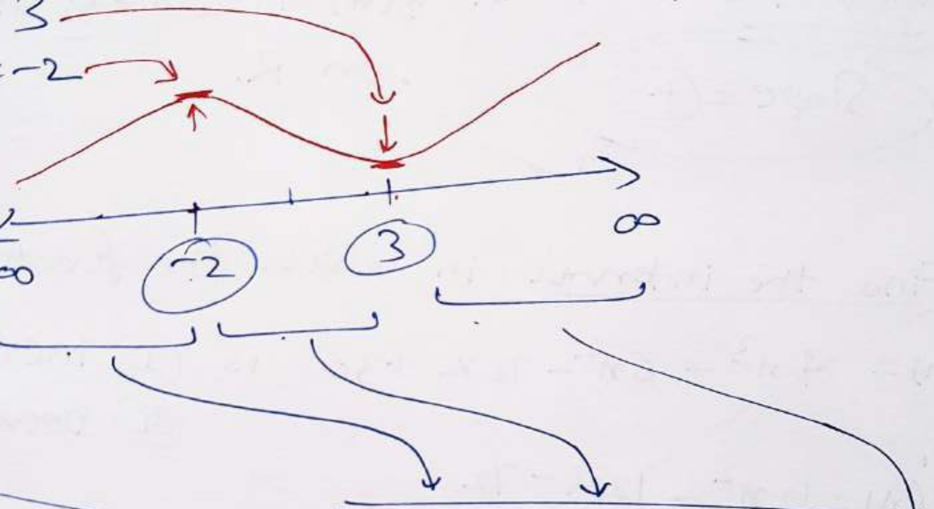
$\Rightarrow f'(x) = 12(x^2 - x - 6)$

$\Rightarrow f'(x) = 12(x^2 - 3x + 2x - 6)$

$\Rightarrow f'(x) = 12(x-3)(x+2) = 0$ (For turning points)

$x = 3$

$x = -2$



$f'(x) = 12(x-3)(x+2)$

(−)(−)	(−)(+)	(+)(+)
+	−	+
Increasing	Decreasing	Increasing

Ⅰ. Increasing Interval → $(-\infty, -2) \cup (3, \infty)$

Ⅱ Decreasing Interval → $(-2, 3)$

[redacted]

Class 12 Maths Application of Derivatives

Exercise - 6.2 Chapter-6

(Increasing and Decreasing Functions)

Q.1 Show that the function given by $f(x) = 3x+17$ is strictly increasing on R.

Solution: $f(x) = 3x+17$

on R

$\downarrow$

from R to R

$\uparrow$ input x $\updownarrow$ output

Strictly Inc.

$\boxed{f'(x) > 0}$ ✓

Inc.

$\boxed{f'(x) \geq 0}$

$x \in R$

$$\boxed{f'(x) = 3} = \oplus ve$$

$$\boxed{f'(x) > 0}, \; x \in R$$

$\therefore$ $f(x) = 3x+17$ is always strictly increasing on $x \in R$.

Q.2 Show that the function given by $f(x) = e^{2x}$ is strictly increasing on R.

Sol^n: $f(x) = e^{2x}$ e = euler's no. = 2.718...

$\Rightarrow f'(x) = e^{2x} \cdot 2$ $(+)^{(\text{Power})}$

$$\boxed{f'(x) = \underbrace{2}_{+}\underbrace{e^{2x}}_{+}} = \oplus ve.$$

$(+)^{+} = \oplus$ ✓

$(+)^{(-)} = \oplus$ ✓

$f'(x) = 2e^{2x} > 0, \; x \in R$

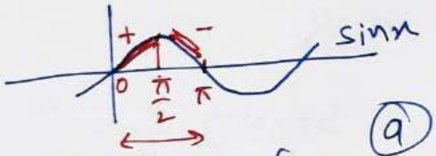
$\therefore$ $f(x)$ is strictly increasing on $x \in R$.

$\boxed{Q.3}$ Show that the function given by $f(x) = \sin x$ is –

ⓐ strictly increasing in $(0, \frac{\pi}{2})$ ⓑ strictly decreasing in $(\frac{\pi}{2}, \pi)$

ⓒ neither increasing nor decreasing in $(0, \pi)$

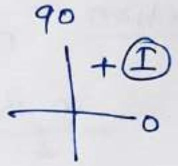
Solution.



$f(x) = \sin x$

$f'(x) = \cos x$

ⓐ In $(0, \frac{\pi}{2})$

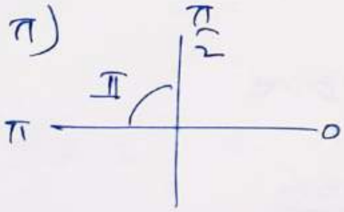


$\cos x > 0$

$f'(x) > 0$

$\therefore f(x)$ is strictly increasing in $(0, \frac{\pi}{2})$.

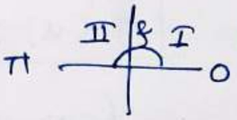
ⓑ In $(\frac{\pi}{2}, \pi)$



$\cos x < 0$

$f'(x) < 0$ ∴ Strictly Decreasing.

ⓒ In $(0, \pi)$



I & II

$\cos x > 0$ Ⓘ

$\cos x < 0$ ⅠⅠ

$\Rightarrow$ कहीं $f'(x) > 0$

कहीं $f'(x) < 0$

∴ neither increasing nor decreasing in $(0, \pi)$

Q. 4 Find the intervals in which the function f given by $f(x) = 2x^2 - 3x$ is $\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$

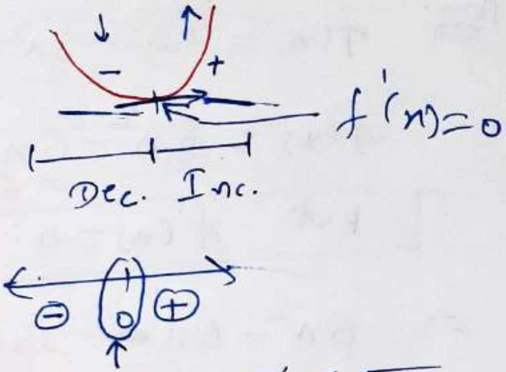
ⓐ strictly increasing ⓑ strictly decreasing.

Ans.

$f(x) = 2x^2 - 3x$

Diff.

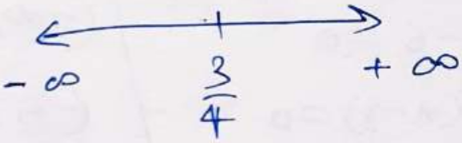
$f'(x) = \underbrace{4x - 3}$



Put $f'(x) = 0$

$\Rightarrow 4x - 3 = 0$

$\Rightarrow \boxed{x = \frac{3}{4}}$



$f'(x) = 4x - 3$ For $\left(-\infty, \frac{3}{4}\right) \rightarrow f'(x) < 0$

(Strictly decreasing)

for $\left(\frac{3}{4}, \infty\right) \rightarrow f'(x) > 0$

Strictly Increasing.

Q.5 Find the Intervals in which the function

$\boxed{f(x) = 2x^3 - 3x^2 - 36x + 7}$ is

ⓐ strictly increasing. ⓑ strictly decreasing.

Ans. $f(x) = 2x^3 - 3x^2 - 36x + 7$

$f'(x) = 6x^2 - 6x - 36$ ✓

$= 6(x-3)(x+2)$

$\boxed{\text{put } f'(x) = 0}$ $x = ?$

$\Rightarrow 6x^2 - 6x - 36 = 0$

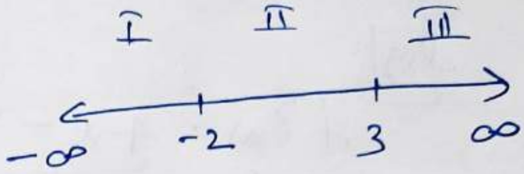
$\Rightarrow x^2 - x - 6 = 0$

$\Rightarrow \underline{x^2 - 3x} + \underline{2x - 6} = 0$

$\Rightarrow x(x-3) + 2(x-3) = 0$

$\Rightarrow (x-3)(x+2) = 0$

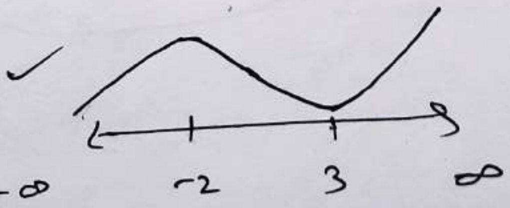
$\boxed{x = 3, -2}$



Interval	$f'(x)$ Sign	↑/↓
$(-\infty, -2)$	$(-).(-) = (+)$	↑
$(-2, 3)$	$(-).(+) = (-)$	↓
$(3, \infty)$	$+.+ = (+)$	↑

ⓐ Strictly increasing, $(-\infty, -2) \cup (3, \infty)$

ⓑ Strictly decreasing $(-2, 3)$



Exercise 6.2 | Chapter 6

Q.6 Find the intervals for strictly increasing or decreasing —

(a) x^2+2x-5

$f(x) = x^2+2x-5$

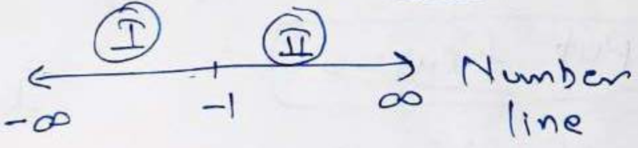
$f'(x) = 2x+2 = 2(x+1)$

Put $f'(x)=0$

$\Rightarrow 2x+2=0$

$\Rightarrow x+1=0$

$\boxed{x=-1}$



Interval	Sign of $f'(x)$	↑/↓
Ⅰ $(-\infty,-1)$	$(-)$	↓ (Dec.)
Ⅱ $(-1,\infty)$	$(+)$	↑ (Inc.)

Strictly Increasing in $(-1,\infty) = x>-1$

Strictly Decreasing in $(-\infty,-1) = x<-1$

(b) $f(x) = 10-6x-2x^2$

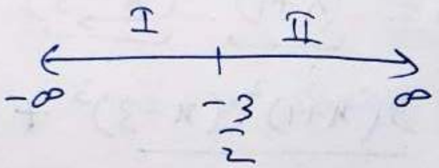
$f'(x) = -6-4x$

$f'(x) = -2(3+2x)$

Put $f'(x)=0$

$\Rightarrow -2(3+2x)=0$

$x = \frac{-3}{2}$



Intervals	Sign of $f'(x)$	↑/↓
$(-\infty,-\frac{3}{2})$	$(+)$	↑ (Increase)
$(-\frac{3}{2},\infty)$	$(-)$	↓ (Decreasing)

St. Increasing

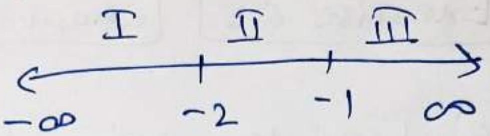
St. Decreasing

ⓒ $f(x) = -2x^3 - 9x^2 - 12x + 1$

$\Rightarrow f'(x) = -6x^2 - 18x - 12$

$\Rightarrow f'(x) = -6(x^2 + 3x + 2)$

$\Rightarrow f'(x) = -6(x+1)(x+2)$



$\boxed{\text{Put } f'(x) = 0}$

$-6(x+1)(x+2) = 0$

$\boxed{x = -1, -2}$

Interval	Sign of $f'(x)$	↑/↓
Ⓘ $(-\infty, -2)$	$(-)$	↓(Dec.)
Ⓘⓘ $(-2, -1)$	$(+)$	↑(Inc.)
Ⓘⓘⓘ $(-1, \infty)$	$(-)$	↓(Dec.)

strictly increasing in $\underline{(-2, -1)}$

strictly Decreasing in $\underline{(-\infty, -2) \cup (-1, \infty)}$

ⓓ $f(x) = 6 - 9x - x^2$ ✓

ⓔ $f(x) = \underline{(x+1)^3} \cdot \underline{(x-3)^3}$ $\boxed{(u.v)' = u'.v + u.v'}$

$f'(x) = \underline{3(x+1)^2 \cdot (x-3)^3} + \underline{(x+1)^3 \cdot 3(x-3)^2}$

$f'(x) = 3(x+1)^2(x-3)^2(\underline{x-3} + \underline{x+1})$

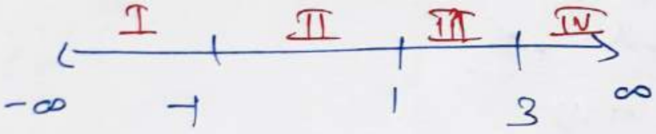
$f'(x) = 3(x+1)^2(x-3)^2(2x-2)$

$f'(x) = 6(x+1)^2(x-3)^2(x-1)$

Now put $f'(x)=0$

$$\Rightarrow 6\underline{(x+1)}^2\underline{(x-3)}^2\underline{(x-1)} = 0$$

$$\underline{x=-1},\ \underline{x=3},\ \underline{x=1}$$



$$\boxed{f'(x) = 6\underset{\oplus}{(x+1)^2}\underset{\oplus}{(x-3)^2}\underset{?}{(x-1)}}$$

Intervals	Sign of $f'(x)$	↑/↓
$(-\infty,-1)$	$(-)$	↓ (Decreasing)
$(-1,1)$	$(-)$	↓ Decreasing
$(1,3)$	$(+)$	↑ (Inc.) ✓
$(3,\infty)$	$(+)$	↑ Increasing! ✓

Strictly increasing in $(1,3)\cup(3,\infty)$

Strictly Decreasing in $(-\infty,-1)\cup(-1,1)$

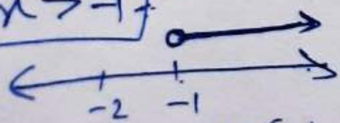
Q.7 Show that $y = \log(1+x) - \frac{2x}{2+x}$, $\boxed{x>-1}$ is an increasing function of x throughout its Domain

Ans. $f(x) = \log(1+x) - \frac{2x}{2+x}$

Domain

$1+x>0$ | $2+x \neq 0$

$\boxed{x>-1}$ | $\boxed{x\neq -2}$ ✓



Domain

$x \in (-1,\infty)$

$$\boxed{\text{Domain } (-1,\infty)}$$

$$f(x) = \log(1+x) - \frac{2x}{(2+x)} \quad \frac{u}{v}$$

$x > -1$

Domain $(-1,\infty)$

$$f'(x) = \frac{1}{1+x} - \left[\frac{(2)(2+x)-2x(1)}{(2+x)^2}\right] \rightarrow \left(\frac{u}{v}\right)' = \frac{u'.v - u.v'}{\boxed{v^2}}$$

$$f'(x) = \frac{1}{1+x} - \left[\frac{4+\cancel{2x}-\cancel{2x}}{(2+x)^2}\right]$$

$$f'(x) = \frac{(2+x)^2 - (1+x).4}{(1+x).(2+x)^2}$$

$$f'(x) = \frac{\cancel{4}+\cancel{4x}+x^2-\cancel{4}-\cancel{4x}}{(1+x)(2+x)^2}$$

$$f'(x) = \frac{\boxed{x^2} \rightarrow \oplus}{\underset{\hookrightarrow \oplus}{(1+x)}.\boxed{(2+x)^2} \rightarrow \oplus} \geq 0$$

Domain

$x \in (-1,\infty)$

$x > -1$

$\underset{\oplus}{\underline{1+x}} > 0$

$f'(x) = \oplus$ve

$$\boxed{f'(x) \geq 0}$$

$\boxed{f(x) \rightarrow \text{increasing}}$ in its domain, Exercise-6.2 Chapter-6

↑ & ↓

Q.8 Find the value of x for which $y = [x(x-2)]^2$ is an increasing function.

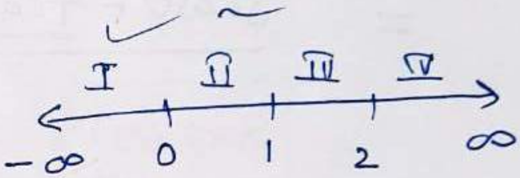
Ans.

$$f(x) = [x(x-2)]^2 = [x^2 - 2x]^2$$

$$f'(x) = 2\underbrace{(x^2-2x)}\cdot\underline{(2x-2)}$$

$$\boxed{f'(x) = 4(x)(x-2)(x-1)}$$

Put $f'(x) = 0$



$\Rightarrow 4(x)(x-2)(x-1) = 0$

$x = 0, 2, 1$

	Interval	Sign of $f'(x)$	↑/↓
I	$(-\infty, 0)$	(−)	Decreasing ↓
II	$(0, 1)$	(+)	↑ Increasing
III	$(1, 2)$	(−)	↓ Decreasing
IV	$(2, \infty)$	(+)	↑ Increasing

Increasing function in $(0,1) \cup (2,\infty)$

Q.9 Prove that $y = \frac{4\sin\theta}{2+\cos\theta} - \theta$ is an increasing function of θ in $\left[0, \frac{\pi}{2}\right]$.

Ans.

$$y = \frac{(4\sin\theta)}{(2+\cos\theta)} - \theta \qquad \boxed{\left(\frac{u}{v}\right)' = \frac{u'.v - u.v'}{v^2}}$$

$$\frac{dy}{d\theta} = \frac{[4\cos\theta]\cdot(2+\cos\theta) - 4\sin\theta.[-\sin\theta]}{(2+\cos\theta)^2} - \frac{1}{1}$$

$$= \frac{8\cos\theta + 4\cos^2\theta + 4\sin^2\theta - (2+\cos\theta)^2}{(2+\cos\theta)^2}$$

$$= \frac{8\cos\theta + \cancel{4(\cos^2\theta + \sin^2\theta)}^{\,1} - \cancel{4} - \cos^2\theta - 4\cos\theta}{(2+\cos\theta)^2}$$

$$\frac{dy}{d\theta} = \frac{4\cos\theta - \cos^2\theta}{(2+\cos\theta)^2}$$

$$\boxed{\frac{dy}{d\theta} = \frac{\cos\theta.(4-\cos\theta)}{(2+\cos\theta)^2}} = \oplus ve$$

$\theta \in \left[0, \frac{\pi}{2}\right]$ I

$\cos\theta = \oplus ve$ ✓

$(4 - \cos\theta) = \oplus ve$ ✓
$\downarrow$
$[-1, 1]$

$(2+\cos\theta)^2 = \oplus ve$ ✓

$\therefore \frac{dy}{d\theta} > 0$ for $\theta \in \left[0, \frac{\pi}{2}\right]$

$\therefore y = \frac{4\sin\theta}{2+\cos\theta} - \theta$ is an increasing function in $\left[0, \frac{\pi}{2}\right]$

Q. 10 Prove that logarithmic function is Strictly increasing on $(0, \infty)$.

Solution $f(x) = \log x$ → $f'(x) = \frac{1}{x}$

$\boxed{\log_e(x)} = \ln(x)$

Natural log.

Domain: $\log\textcircled{x}$

$x > 0$

$x \in (0, \infty)$

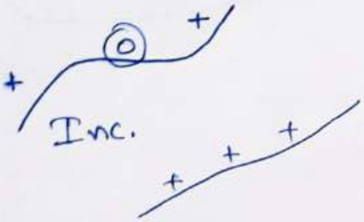
$\boxed{x = \oplus ve}$

Here $x > 0$

$\Rightarrow 0 < \boxed{x} < \infty$

$\Rightarrow \infty > \boxed{\frac{1}{x}} > 0$ (by Reprocating)

$\underbrace{\frac{1}{x}}_{\oplus ve} > 0 \Rightarrow \boxed{f'(x) > 0}$



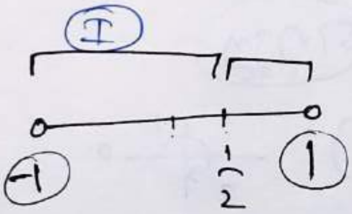
Strictly Inc

$\therefore$ $\log x$ is strictly increasing in $(0, \infty)$

Q.11 Prove that $f(x) = x^2 - x + 1$ is neither strictly increasing nor strictly decreasing on $(-1, 1)$.

Ans. $f(x) = x^2 - x + 1$

$\boxed{f'(x) = 2x - 1}$



Put $f'(x) = 0$

$\Rightarrow 2x - 1 = 0$

$\Rightarrow \boxed{x = \frac{1}{2}}$

Interval	Sign of $f'(x)$	↑/↓
$(-1, \frac{1}{2})$	$(-)$	↓
$(\frac{1}{2}, 1)$	$\oplus$	↑

$\therefore f'(x) > 0, \forall x \in (\frac{1}{2}, 1)$ & $f'(x) < 0$ $\forall x \in (-1, \frac{1}{2})$

$\therefore f(x)$ is ~~neither~~ $(-1, 1)$

Q. 12 Which of the following functions are strictly decreasing on $\left(0, \frac{\pi}{2}\right)$? $x \in \left(0, \frac{\pi}{2}\right)$

✓(A) cosx ✓(B) cos2x ~~(C)~~ cos3x ~~(D)~~ tanx.

Ans

(A) $f(x) = \cos x$

$f'(x) = \underbrace{-}_{(-)}\underbrace{\sin x}_{(+)}$

$f'(x) = \ominus ve$

f(x) strictly Decreasing.

(B) $f(x) = \cos 2x$

$f'(x) = -2\sin(\underline{2x})$

$x \in \left(0, \frac{\pi}{2}\right)$ $\quad$ $0 < x < \frac{\pi}{2}$

$\boxed{\underline{2x}} \in (0, \pi)$ $\quad$ $0 < \underline{2x} < \pi$

$\sin 2x \in (0, 1)$

$\sin 2x > 0$

$\underbrace{-2}_{\ominus}\underbrace{\sin 2x}_{\oplus} < 0$ $\quad$ $\therefore f'(x) < 0$

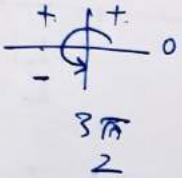
Strictly Dec.

(C) $f(x) = \cos 3x$

$f'(x) = \underbrace{-3}\boxed{\sin 3x}$

$x \in \left(0, \frac{\pi}{2}\right)$

$3x \in \left(0, \frac{3\pi}{2}\right)$



$\sin 3x > 0$ for some values of x

$\underline{\sin 3x < 0}$ for some values of x

कहीं $f'(x) > 0$, कहीं $f'(x) < 0$

(D) $f(x) = \tan x$

$f'(x) = \sec^2 x$

$= (\sec x)^2 > 0$

Increasing

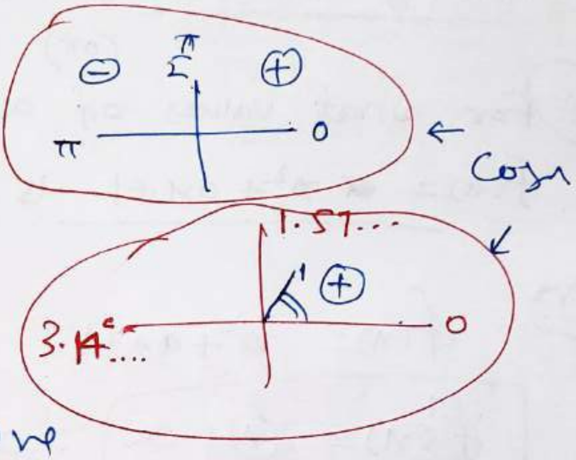
Q.13 On which of the following intervals

$f(x) = x^{100} + \sin x - 1$ strictly decreasing? $\rightarrow$ $f'(x)$ $\ominus$ve

Ⓐ $(0, 1)$ ~~Ⓑ $\left(\frac{\pi}{2}, \pi\right)$~~ ~~Ⓒ $\left(0, \frac{\pi}{2}\right)$~~ Ⓓ None of these

Ans $f(x) = x^{100} + \sin x - 1$

$$\boxed{f'(x) = \underbrace{100 \cdot x^{99}} + \underbrace{\cos x}}$$



Option Ⓐ $x \in (0, 1)$

$$f'(x) = \underbrace{100 \cdot x^{99}}_{\oplus} + \underbrace{\cos x}_{\oplus} = \oplus ve$$

$f'(x) > 0$ (increasing)

Option Ⓑ $x \in \left(\frac{\pi}{2}, \pi\right)$ $x \in (1.57\ldots, 3.14\ldots)$

1.58

$$\boxed{f'(x) = \underbrace{100 \cdot x^{99}} + \underline{\cos x}}$$

$\uparrow\uparrow\uparrow$ (बहुत बड़ा $\oplus$ve no.) $\ominus$ (-1) $= \oplus ve$ $f'(x) > 0$

Increasing

Option Ⓒ $x \in \left(0, \frac{\pi}{2}\right)$

$$f'(x) = \underset{\oplus}{100 \cdot x^{99}} + \underset{\oplus}{\cos x} = \oplus ve$$

(Increasing)

Exercise 6.2   Chapter ⑥

Q.14 Find the least value of a such that the function f given by $f(x) = x^2+ax+1$ is strictly increasing on $(1,2)$. (pdf)

(or)

(Book) For what values of a the function f given by $f(x) = x^2+ax+1$ is ~~ine~~ increasing on $[1,2]$?

Ans.

$$f(x) = x^2+ax+1 \qquad \boxed{f'(x) > 0}$$

$$\boxed{f'(x) = 2x + a} > 0$$

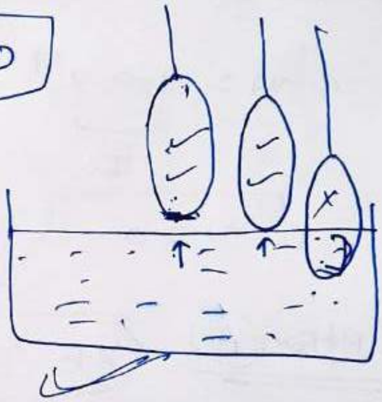
$$x \in (1,2)$$

$$\boxed{1 < x < 2}$$

$$\Rightarrow 2 < 2x < 4$$

$$\Rightarrow 2+a < 2x+a < 4+a$$

$$0 \leq \boxed{2+a < f'(x) < 4+a}$$



$$\because f'(x) > 0$$

$$\therefore \text{minimum value of } f'(x) \geq 0$$

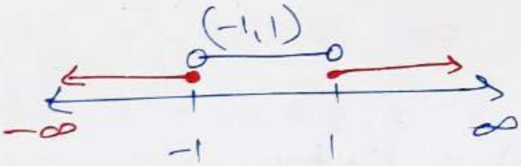
$$\Rightarrow 2+a \geq 0$$

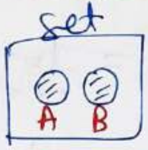
$$\Rightarrow \boxed{a \geq -2}$$

least value of $a = -2$

Q.15 Let I be any interval disjoint from $(-1,1)$. Prove that the function f given by $f(x) = x + \frac{1}{x}$ is strictly increasing on I.

Ans.





$\boxed{A \cap B = \phi}$

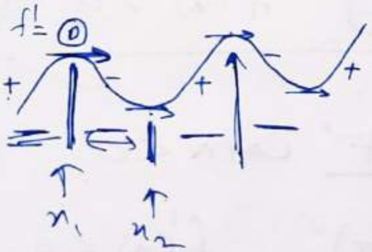
Disjoint

$$\boxed{I \in (-\infty,-1] \cup [1,\infty)}$$

$$f(x) = x + \frac{1}{x}$$

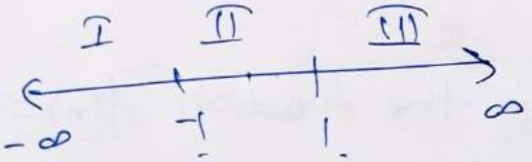
$$f'(x) = 1 - \frac{1}{x^2} = 1^2 - \left(\frac{1}{x}\right)^2 = \left(1+\frac{1}{x}\right)\left(1-\frac{1}{x}\right)$$

$\boxed{f'(x)=0}$ put



$$\left(1+\frac{1}{x}\right)\left(1-\frac{1}{x}\right)=0$$

$\boxed{x=-1}$ $\boxed{x=1}$



$$f'(x) = \left(1+\frac{1}{x}\right)\left(1-\frac{1}{x}\right)$$

Interval	Sign of $f'(x)$	↑/↓
I $(-\infty,-1)$	+	↑ (Inc.)
II $(-1,1)$	−	↓ (Dec.)
III $(1,\infty)$	+	↑ Increa

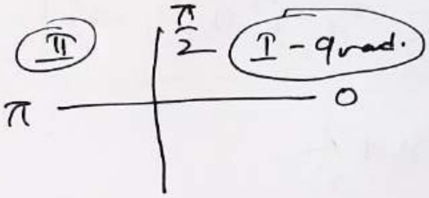
∴ $f(x) = x + \frac{1}{x}$ is strictly increasing in $(-\infty,-1) \cup (1,\infty)$ I

Q.16 Prove that the function $\boxed{f(x) = \log \sin x}$ is strictly increasing on $\left(0, \frac{\pi}{2}\right)$ and strictly decreasing on $\left(\frac{\pi}{2}, \pi\right)$.

Ans. $f(x) = \underset{\rightarrow}{\log}\ \underset{\rightarrow}{\sin x}$

$$f'(x) = \frac{1}{\sin x} \cdot (\cos x) = \cot x$$

$$\boxed{f'(x) = \cot x}$$



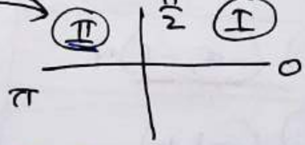
In I - quadrant $\cot x > 0$

$x \in \left(0, \frac{\pi}{2}\right)$ $f'(x) > 0$ (Strictly Inc.) ✓

In II - quadrant $\cot x < 0$

$x \in \left(\frac{\pi}{2}, \pi\right)$, $f'(x) < 0$ (Strictly Decreasing)

Q.17 Prove that the function $f(x) = \log \cos x$ is strictly decreasing on $\left(0, \frac{\pi}{2}\right)$ and strictly increasing on $\left(\frac{\pi}{2}, \pi\right)$ →



Ans. $f(x) = \underset{\rightarrow}{\log}\ \underset{\rightarrow}{\cos x}$

$$f'(x) = \frac{1}{\cos x} \cdot (-\sin x)$$

$$\boxed{f'(x) = -\underset{\oplus}{\tan x}}$$

I-quadrant $x \in \left(0, \frac{\pi}{2}\right)$

$\tan x > 0$

$-\tan x < 0$ $\therefore f'(x) < 0$

Strictly Decreasing

II-quad. $x \in \left(\frac{\pi}{2}, \pi\right)$, $\tan x < 0$ $f'(x) > 0$ Strictly Inc.

$-\tan x > 0$

Q. 18 Prove that the function given by $f(x) = x^3 - 3x^2 + 3x - 100$ is increasing in R.

Ans. $x \in R$.

$f(x) = x^3 - 3x^2 + 3x - 100$

$f'(x) = 3x^2 - 6x + 3$

$f'(x) = 3(\underline{x^2 - 2x + 1})$

$f'(x) = 3(x-1)^2$

$(x-1)^2$ Perfect Square ≥ 0

For $x \in R$

$\underbrace{3(x-1)^2}$ ≥ 0

$\boxed{f'(x) \geq 0} \longrightarrow$ $f(x)$ is increasing ✓ for $x \in R$.

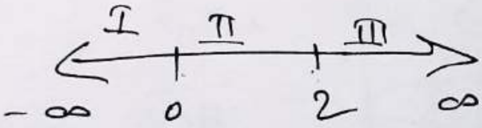
Q.19 The interval in which $y = x^2 \cdot e^{-x}$ is increasing is —

(A) $(-\infty, \infty)$ (B) $(-2, 0)$ (C) $(2, \infty)$ [✓ (D) $(0, 2)$]

Ans. $y = \underbrace{x^2} \cdot \underbrace{e^{-x}}$

$f'(x) = \underbrace{2x \cdot e^{-x}} - \underbrace{x^2 \cdot e^{-x}}$

$f'(x) = \underbrace{x \cdot (2-x)} \cdot \underbrace{e^{-x}}$



$\boxed{\text{Put } f'(x) = 0}$

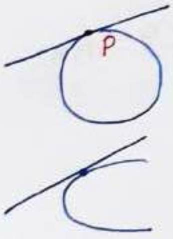
$x(2-x) \cdot e^{-x} = 0$

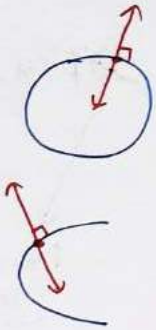
$x = 0$ ↗ $x = 2$ ↗ $e^{-x} \neq 0$

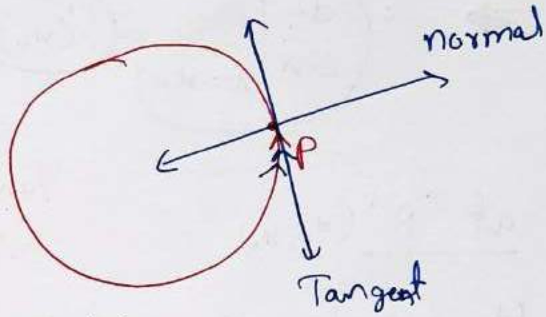
Interval	Sign of $f'(x)$	↑/↓
I $(-\infty, 0)$	$(-)$	↓ (Dec.)
II $(0, 2)$	$(+)$	↑ (Inc.)
III $(2, \infty)$	$(-)$	↓ Dec.

Tangents and Normals

[स्पर्श रेखा] [अभिलम्ब]







At point 'p'

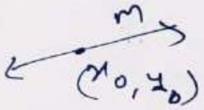
$\boxed{\text{Graph} \parallel \text{Tangent}}$

$\boxed{\text{Normal} \perp \text{Tangent}}$

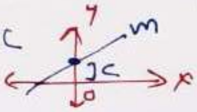
$\boxed{\text{Normal} \perp \text{Graph}}$

Revision:

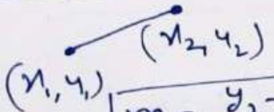
↳ Line $(y-y_0)=m(x-x_0)$



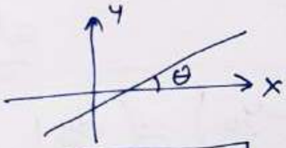
$y = mx+c$



Slope (m)



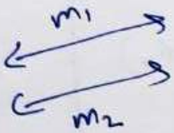
$\boxed{m = \frac{y_2-y_1}{x_2-x_1}}$



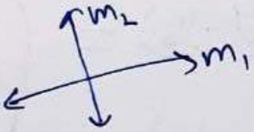
$\boxed{m = \tan\theta}$

$m = \frac{dy}{dx}$

Diff.

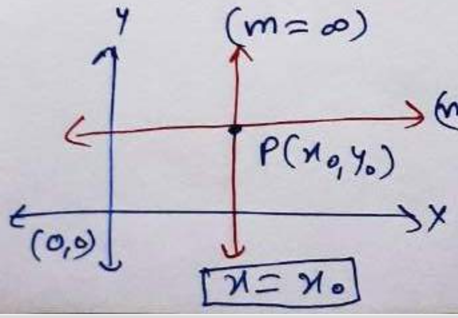


Parallel $\boxed{m_1 = m_2}$



Perpendicular $\boxed{m_1 m_2 = -1}$

#



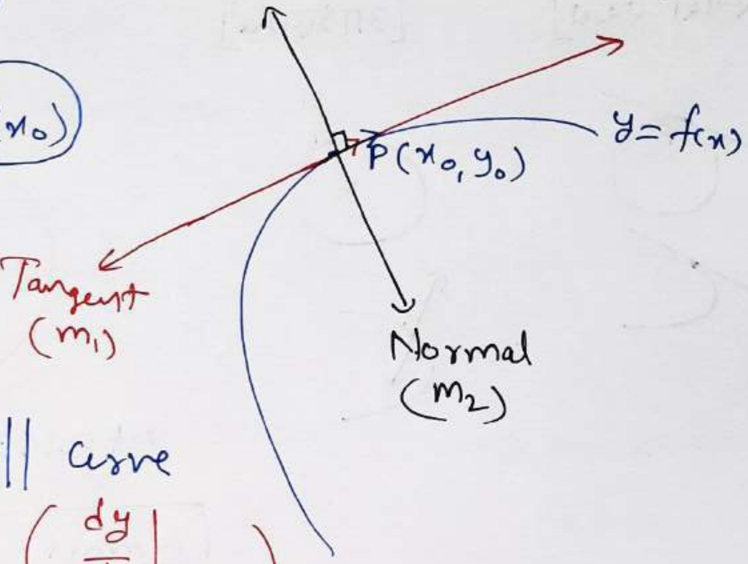
$\boxed{y = y_0}$

$\boxed{x = x_0}$

Equations of Tangents and Normals :→

Slope of Curve $y=f(x)$
at $x=x_0$

$$\Rightarrow = \left.\frac{dy}{dx}\right|_{x=x_0} = f'(x_0)$$



at 'p' (x_0,y_0)

$$\underbrace{\underset{(m_2)}{\text{Normal}} \perp \underset{(m_1)}{\text{Tangent}}}_{m_1 m_2=-1} \parallel \underset{\left(\left.\frac{dy}{dx}\right|_{x=x_0}\right)}{\text{curve}}$$

$$\boxed{m_1 = \left.\frac{dy}{dx}\right|_{x=x_0}}$$

$$m_2 = \frac{-1}{m_1}$$

$$\boxed{m_2 = -\frac{1}{\left.\frac{dy}{dx}\right|_{x=x_0}}}$$

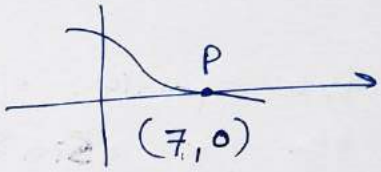
at Point $P(x_0,y_0)$

Tangent → $\boxed{(y-y_0) = \left(\left.\frac{dy}{dx}\right|_{x=x_0}\right)\cdot(x-x_0)}$

Normal → $\boxed{(y-y_0) = \left(-\frac{1}{\left.\frac{dy}{dx}\right|_{x=x_0}}\right)(x-x_0)}$

e.g Find the slope of the tangent and normal to the curve $y = \frac{x-7}{(x-2)(x-3)}$ at the point where it cuts the x-axis. Hence find equations of tangent and normal.

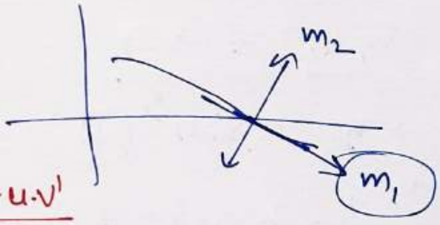
Ans. $y = \frac{(x-7)}{(x-2)(x-3)}$



For the point P on x-axis: $y = 0 \Rightarrow 0 = \frac{(x-7)}{(x-2)(x-3)} \Rightarrow 0 = (x-7)$ $(x = 7)$

Point P(7,0)

$y = \frac{(x-7)}{x^2-5x+6}$ $\left(\frac{u}{v}\right)'$ $\searrow$ $\frac{u'.v - u.v'}{v^2}$



$$\frac{dy}{dx} = \frac{1(x^2-5x+6) - (x-7).(2x-5)}{(x^2-5x+6)^2}$$

$$\left.\frac{dy}{dx}\right|_{(7,0)} = \frac{\cancel{(49-35+6)} - \overset{0}{\cancel{(7-7)}}(2x-5)}{(49-35+6)^{\cancel{2}}} = \frac{1}{20}$$

$\left.\frac{dy}{dx}\right|_{(7,0)} = \frac{1}{20}$ = Slope of tangent

-20 = Slope of normal $\boxed{\frac{1}{m_1 m_2 = -1}}$

Point (7, 0)

Equation of Tangent (7,0) $(y-\underset{0}{y_0}) = m(x-\overset{7}{x_0})$

$$(y-0) = \left(\frac{1}{20}\right)\cdot(x-7)$$

Equation of Normal.

$$(y-0) = (-20)\cdot(x-7)$$

e.g. Find the equation of tangent to the curve given by $\boxed{x = a\sin^3 t}$, $\boxed{y = b\cos^3 t}$ at a point $t = \frac{\pi}{2}$.

(Parametric Form)

$$\boxed{\text{Eq}^n\text{. of Tangent } (y-y_0) = \underbrace{\left.\frac{dy}{dx}\right|_{(x_0,y_0)}}_{0}\cdot(x-x_0)}$$

Point $(x_0, y_0) \equiv (a, 0)$

$\boxed{t=\frac{\pi}{2}} \rightarrow x_0 = a\sin^3\left(\frac{\pi}{2}\right) = a$ Point $(a,0)$

$\rightarrow y_0 = b\cos^3\left(\frac{\pi}{2}\right) = 0$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\frac{d(b\cos^3 t)}{dt}}{\frac{d(a\sin^3 t)}{dt}} = \frac{b\cdot\cancel{3}(\cos t)^{\cancel{2}}\cdot(-\cancel{\sin t})}{a\cdot\cancel{3}(\sin t)^{\cancel{2}}\cdot(\cancel{\cos t})}$$

$$\frac{dy}{dx} = -\frac{b(\cot t)}{a}$$

$$\left.\frac{dy}{dx}\right|_{(a,0)} = -\frac{b}{a}\left(\cot\frac{\pi}{2}\right) = 0 = \text{Slope of tangent} \quad \boxed{t=\frac{\pi}{2}}$$

Equation of Tangent $(y-y_0) = m(x-x_0)$

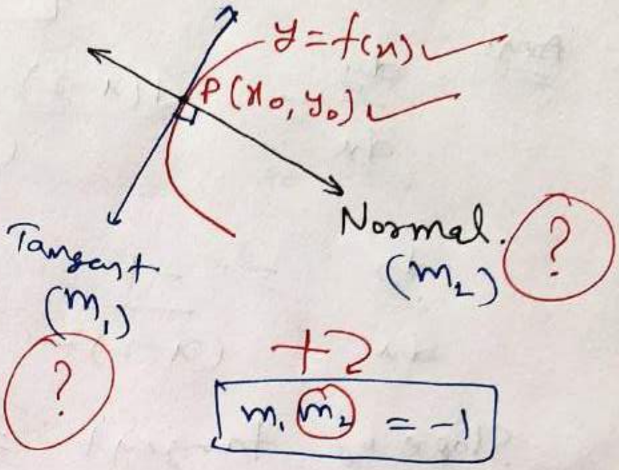
(a,0) $(y-0) = 0(x-a)$

$$\boxed{y-0=0} \Rightarrow \boxed{y=0}$$

$\boxed{\text{Exercise 6.3}}$ Chapter 6

Tangents and Normals.

At Point 'P' (x_0, y_0)



$$\boxed{m_1 \boxed{m_2} = -1}$$

$$\boxed{\text{Tangent} \parallel \text{Curve} \perp \text{Normal}}$$

$$m_1 = \frac{dy}{dx}\Big|_{(x_0,y_0)} \qquad \boxed{m_2}$$

$$\boxed{\text{Tan}}\ m_1 = \frac{dy}{dx}\Big|_{(x_0,y_0)}$$

$$m_2 = \frac{-1}{\left(\frac{dy}{dx}\right)\Big|_{(x_0,y_0)}}$$

Exercise 6.3

$\boxed{Q.1}$ Find the slope of the tangent to the curve $y = 3x^4 - 4x$ at $x=4$.

Ans: Slope of curve $= \frac{dy}{dx} = 12x^3 - 4$

Slope of curve at $(x=4) = \frac{dy}{dx}\Big|_{x=4} = 12(4)^3 - 4$

$= 12 \times 64 - 4 = 764$

Slope of tangent at $(x=4) = 764$

$\boxed{Q.2}$ Find the slope of the tangent to the Curve ~~to the~~ $\underbrace{y = \frac{x-1}{x-2}}$, $x \neq 2$ at $\underbrace{x=10}$

Ans.

$$\frac{dy}{dx} = \frac{1(x-2) - (x-1).(1)}{(x-2)^2} = \frac{\cancel{x}-2 - \cancel{x}+1}{(x-2)^2}$$

$$\frac{dy}{dx} = \frac{-1}{(x-2)^2} \quad \text{(Curve)}$$

Slope of tangent at $x=10$ = Slope of curve at $x=10$ $= \left.\frac{-1}{(x-2)^2}\right|_{x=10}$

$$= \frac{-1}{8^2} = \frac{-1}{64} ✓$$

$\boxed{Q.3}$ Find the slope of the tangent to the curve $y = x^3 - x + 1$ at the point whose x-coordinate is 2. $\boxed{x=2}$

Ans.

$$\frac{dy}{dx} = 3x^2 - 1$$

Slope of tangent (at $x=2$) = Slope of Curve (at $x=2$) $= \left.3x^2 - 1\right|_{x=2}$

$$= 3(2^2) - 1$$

$$= 11 ✓$$

Q.4 Find the slope of the tangent to the curve $y = x^3 - 3x + 2$ at the point whose x-coordinate is 3.

Same as Q1, Q2, Q3

Q.5 Find the slope of the normal to the curve $\underbrace{x = a\cos^3\theta, \ y = a\sin^3\theta}_{\text{(Parametric form.)}}$ at $\underbrace{\theta = \frac{\pi}{4}}$.

Ans. $$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{\left(\frac{d(a\sin^3\theta)}{d\theta}\right)}{\left(\frac{d(a\cos^3\theta)}{d\theta}\right)} = \frac{\cancel{a}.\cancel{3}\sin^{\cancel{2}}\theta . \cancel{\cos\theta}}{\cancel{a}.\cancel{3}\cos^{\cancel{2}}\theta . (-\cancel{\sin\theta})}$$

$$\frac{dy}{dx} = -\frac{\sin\theta}{\cos\theta} = -\tan\theta$$

$$\left.\frac{dy}{dx}\right|_{\theta = \frac{\pi}{4}} = -\tan\left(\frac{\pi}{4}\right) = -1 = m_1$$

$\uparrow$

(Slope of tangent) = (Slope of curve)

Normal ⊥ Tangent
(m_2) (m_1)

$$m_2 . m_1 = -1$$

$$m_2 = -\frac{1}{m_1}$$

Slope of normal $= -\frac{1}{\left(\text{Slope of tangent}\right)} = -\frac{1}{(-1)}$

$= 1$ ✓

Q.6 Find the slope of the normal to the curve $\underbrace{x = 1 - a\sin\theta, \; y = b\cos^2\theta}_{\text{(Parametric Form)}}$ at $\underbrace{\theta = \frac{\pi}{2}}_{\text{(Point)}}$.

Ans.

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{\frac{d(b\cos^2\theta)}{d\theta}}{\frac{d(1 - a\sin\theta)}{d\theta}} = \frac{b.2\cancel{\cos\theta}.(+\sin\theta)}{(0 + a\cancel{\cos\theta})}$$

$$\frac{dy}{dx} = \frac{2b\sin\theta}{a} = \text{Slope of curve} = \text{Slope of tangent.}$$

At $\theta = \frac{\pi}{2}$ Slope of tangent $(m_1) = \frac{2b\left(\sin\frac{\pi}{2}\right)}{a}$

$$= \boxed{\frac{2b}{a}}$$

Slope of Normal (at $\theta = \frac{\pi}{2}$) $= -\frac{1}{m_1} = -\frac{1}{\left(\frac{2b}{a}\right)} = -\frac{a}{2b}$ ✓

Q.7 Find the points (x, y) at which the tangent to the curve $y = x^3 - 3x^2 - 9x + 7$ is parallel to the x-axis.

Ans. If tangent is parallel to the x-axis then slope of tangent $= 0$ ✓

$$y = x^3 - 3x^2 - 9x + 7$$

$$\text{Slope of tangent} = \frac{dy}{dx} = 3x^2 - 6x - 9 = 0 \text{ (put) (Given)}$$

$\frac{dy}{dx} = \underbrace{3x^2 - 6x - 9}{} = 0$

$\Rightarrow 3(x^2 - 2x - 3) = 0$

$\Rightarrow x^2 - 3x + x - 3 = 0$

$\Rightarrow x(x-3) + 1(x-3) = 0$

$\Rightarrow (x-3)(x+1) = 0$

$x = 3, \ x = -1$

$y = x^3 - 3x^2 - 9x + 7$

$\boxed{x = 3}$

$y = \cancel{3^3} - \cancel{3(3)^2} - 9 \times 3 + 7$

$\boxed{y = -20}$

Point $(3, -20)$

$\boxed{x = -1} \Rightarrow y = (-1)^3 - 3(-1)^2 - 9(-1) + 7$

$\Rightarrow y = -1 - 3 + 9 + 7 = 12$

Point $(-1, 12)$

$\boxed{Q.8}$ Find $\overset{(x,y)}{a}$ point on the curve $y = (x-2)^2$ at which tangent is parallel to the chord joining the points $(2,0)$ and $(4,4)$

Ans. Slope of Chord $= \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-0}{4-2} = \frac{4}{2} = 2$

chord $\parallel$ tangent $(m_1 = m_2) \Rightarrow$ $\boxed{\text{Slope of tangent} = 2}$ — ②

$y = (x-2)^2$

Slope of tangent $= \frac{dy}{dx} = 2(x-2)$ — ①

$2(x-2) = 2$

$\Rightarrow x - 2 = 1$

$\Rightarrow \boxed{x = 3}$

$y = (x-2)^2$

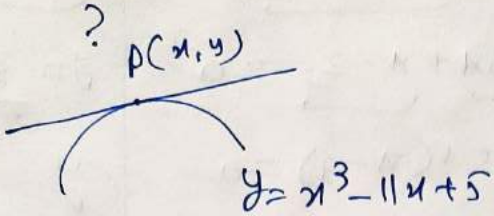
$y = (3-2)^2 = 1$

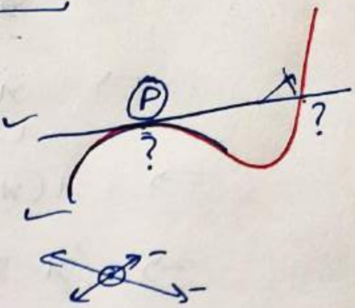
$\boxed{\text{Point } (3, 1)}$

Q.9 Find the point on the curve $\boxed{y=x^3-11x+5}$ at which the tangent is $\underline{y=x-11}$.

Ans.

$y=x-11$
(Tangent)





Line $\boxed{y=mx+c}$ (Slope of this line)

$y=x-11$ tangent

$\Downarrow$

$y=1.x-11$

Slope of tangent $=1$ ——①

$y=x^3-11x+5$

Slope of tangent $=\frac{dy}{dx}$

$=3x^2-11$ ——②

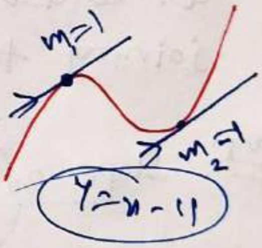
By eqn ① & ② →

$1=3x^2-11$

$\Rightarrow 12=3x^2$

$\Rightarrow 4=x^2$

$\rightarrow \boxed{x=\pm 2}$



$x=2$ $\quad$ $x=-2$

$y=x^3-11x+5$ Curve

$y=2^3-11(2)+5$

$y=8-22+5$

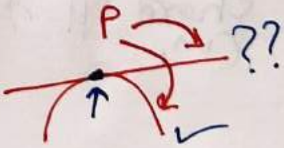
$y=-9$

Point $(2,-9)$

$y=-8+22+5$

$y=19$

$(-2,19)$ Point



Tangent $y=x-11$

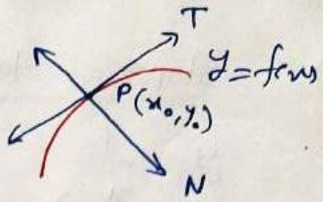
$(2,-9)$ $\quad -9=2-11$

$(-2,19)$ $\quad 19\neq -2-11$

$19\neq -13$

Exercise 6.3   Chapter-6   (Tangents & Normals)

Q.10 Find the equation of all lines having slope -1 that are tangents to the curve $y = \frac{1}{x-1}$, $x \neq 1$.



Curve ∥ Tangent ⊥ Normal

$m_c = m_T$ m_N

$\left.\frac{dy}{dx}\right|_{(x_0,y_0)}$ $m_N = \frac{-1}{m_T}$

Ans. Line → Point (x_0, y_0) ?
→ Slope (m) ✓

$(y - y_0) = m(x - x_0)$

$y = \frac{1}{x-1}$ by diff. $\boxed{\frac{dy}{dx} = -\frac{1}{(x-1)^2}}$ = Slope of tangent $= m_T$

Given Slope of tangent $= -1$ ✓ →

$-1 = \frac{-1}{(x-1)^2}$

$\Rightarrow (x-1)^2 = 1$

$\Rightarrow (x-1) = \pm 1 \Rightarrow x = 1 \pm 1$

$\boxed{x = 2, \; x = 0}$ x-coordinates

$y = \frac{1}{x-1}$ $x=2 \rightarrow y = \frac{1}{2-1} = 1$ $(2,1)$

$y = \frac{1}{x-1}$ $x=0 \rightarrow y = \frac{1}{0-1} = -1$ $(0,-1)$

$m = -1$, Point $(2,1)$ $(y-1) = -1(x-2) \Rightarrow \boxed{y = -x+3}$ ✓

$m = -1$, Point $(0,-1)$ $(y+1) = -1(x-0) \Rightarrow y+1 = -x$

$\boxed{y + x + 1 = 0}$ ✓

Q.11 Find the equation of all lines having slope 2 which are tangents to the curve $y=\frac{1}{x-3}, x\neq 3$.

Ans. Slope of tangent = 2 (given) ✓ —— ①

$$y=\frac{1}{x-3}$$

Slope of tangent $=\frac{dy}{dx}=-\frac{1}{(x-3)^2}$ ✓ —— ②

By eqⁿ ① & ②

$$-\frac{1}{(x-3)^2}=2$$

$$\Rightarrow \left(-\frac{1}{2}\right)=(x-3)^2$$

$$\Rightarrow \pm\underbrace{\sqrt{-\frac{1}{2}}}_{\text{imaginary No.}}=(x-3)$$

Not possible.

Q.12 Find the equations of all lines having slope '0' which are tangent to the curve $y=\frac{1}{(x^2-2x+3)}$.

Ans. Slope of tangent = 0 (given) —— ①

Slope of tangent $=\frac{dy}{dx}=-\frac{1}{(x^2-2x+3)^2}\cdot(2x-2)$ —— ②

By eqⁿ ① & ②:

$$\Rightarrow -\frac{2(x-1)}{(x^2-2x+3)^2}=0$$

$$\Rightarrow x-1=0$$

$$\boxed{x=1}$$

$$y=\frac{1}{x^2-2x+3}$$

Put $x=1$

$$y=\frac{1}{1-2+3}=\frac{1}{2}$$

$$\left(1,\frac{1}{2}\right)$$

$\left(1,\frac{1}{2}\right)$, $m=0$

$$(y-y_0)=m(x-x_0)$$

$$\boxed{y-\frac{1}{2}=0}$$

Tangent

Q.13 Find points on the curve $\boxed{\frac{x^2}{9}+\frac{y^2}{16}=1}$ at which the tangents are (i) Parallel to x-axis.

(ii) Parallel to y-axis.

Ans. $\frac{x^2}{9}+\frac{y^2}{16}=1$

by diff. w.r.t. 'x'

$\Rightarrow \frac{\cancel{2}x}{9}+\frac{\cancel{2}y}{16}\cdot\boxed{\frac{dy}{dx}}=0$

$\Rightarrow \frac{dy}{dx}=-\frac{16}{y}\frac{x}{9}$

$\Rightarrow \boxed{\frac{dy}{dx}=-\frac{16x}{9y}}=m_T$

(i) tangents || x-axis.

$m_T=0$

$\Rightarrow \frac{-16x}{9y}=0$

$\Rightarrow \boxed{x=0}$

x-coordinate,

$\frac{x^2}{9}+\frac{y^2}{16}=1$

put $x=0$

$\frac{y^2}{16}=1$

$\boxed{y=\pm4}$

Points $(0,\pm4)$ ✓

(ii) tangents || y-axis

Slope of tangents $=\infty$

$m_T=\infty=\frac{1}{0}$

$\Rightarrow \frac{-16x}{9y}=\frac{1}{0}$

$\Rightarrow 0=9y$

$\Rightarrow \boxed{y=0}$

y-coordinate

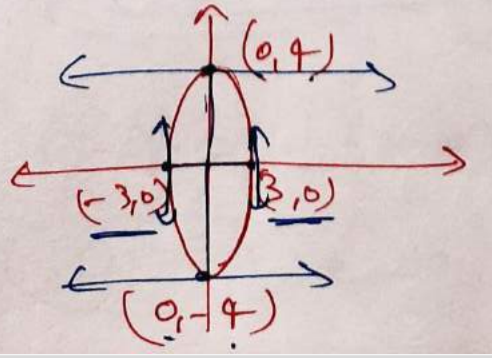
$\frac{x^2}{9}+\frac{y^2}{16}=1$

Put $y=0$

$\Rightarrow \frac{x^2}{9}=1$

$\Rightarrow \boxed{x=\pm3}$

Points $(\pm3,0)$



$\frac{x^2}{9}+\frac{y^2}{16}=1$

Q.14 Find the equations of Tangents and Normals.

(i) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at $(0,5)$ ✓

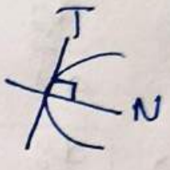
(ii) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at $(1,3)$

$$\text{Slope of tangent} = \frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10$$

at $(0,5)$

$$\text{Slope of tangent at } (0,5) = m_T = 0 - 0 + 0 - 10$$

$$\boxed{m_T = -10}$$



$$m_N \cdot m_T = -1$$

$$m_N = -\frac{1}{m_T} = \frac{-1}{-10} = \frac{1}{10}$$ ✓

$$\boxed{(y - y_0) = m(x - x_0)}$$

Equation of tangent at $(0,5)$

$$(y - 5) = (-10)(x - 0) \Rightarrow y - 5 = -10x$$

$$\Rightarrow \boxed{10x + y = 5}$$ ✓

Equation of Normal at $(0,5)$

$$(y - 5) = \frac{1}{10}(x - 0)$$

$$\Rightarrow 10y - 50 = x$$

$$\Rightarrow \boxed{0 = x - 10y + 50}$$ ✓

(iii) $\boxed{y = x^3}$ at (1,1)

(iv) $\boxed{y = x^2}$ at (0,0)
Not Yourself

$$\frac{dy}{dx} = 3x^2$$

Slope of tangent at $(\overset{x}{1},\overset{y}{1}) = m_T = 3(1)^2 = 3$ ✓

Slope of normal at $(1,1) = m_N = -\frac{1}{3}$ ✓

Equation (1,1)

Tangent. $(y-1) = 3(x-1) \Rightarrow y-1 = 3x-3 \Rightarrow \boxed{y = 3x-2}$

Normal $(y-1) = -\frac{1}{3}(x-1) \Rightarrow 3y-3 = -x+1 \Rightarrow \boxed{x+3y-4=0}$ ✓

(iv) $y = x^2$ at (0,0)

$\frac{dy}{dx} = 2x$, Tangent's slope $\boxed{m_T = 0}$ ↔ $y=k$

Normal's slope $m_N = -\frac{1}{0} = \infty, -\infty$

↕ $x=k$

Equation, $(\underset{0}{y}-\underset{0}{y_0}) = m(x-\underset{0}{x_0})$

Tangent, $m_T = 0$ $\quad y-0 = 0(x-0)$

$y-0=0 \Rightarrow y=0$

Normal, $m_N = \frac{1}{0}$ $\quad (y-0) = \frac{1}{0}(x-0)$

$0 = x$

(V) $x = \cos t$, $y = \sin t$ at $t = \frac{\pi}{4}$

Parametric Form

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d(\sin t)}{dt}}{\frac{d(\cos t)}{dt}} = \frac{\cos t}{-\sin t} = -\cot t$$

Slope of tangent at $t = \frac{\pi}{4} = m_T = -\cot(t)$

$= -\cot\left(\frac{\pi}{4}\right)$

$= -1$

$m_N = \frac{-1}{-1} = +1$

Point $(x_0, y_0) \equiv \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$$x_0 \downarrow \cos\frac{\pi}{4} \downarrow \frac{1}{\sqrt{2}} \qquad y_0 \downarrow \sin\frac{\pi}{4} \downarrow \frac{1}{\sqrt{2}}$$

Equation: $(y - \overset{\frac{1}{\sqrt{2}}}{y_0}) = m(x - \overset{\frac{1}{\sqrt{2}}}{x_0})$

Tangent

$$\left(y - \frac{1}{\sqrt{2}}\right) = -1\left(x - \frac{1}{\sqrt{2}}\right)$$

$\Rightarrow y - \frac{1}{\sqrt{2}} = -x + \frac{1}{\sqrt{2}}$

$\Rightarrow x + y - \frac{2}{\sqrt{2}} = 0$

$\Rightarrow \boxed{x + y - \sqrt{2} = 0}$ ✓

$$\frac{2}{\sqrt{2}} = \frac{\sqrt{2}\cdot\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}}$$

Normal: $y - \frac{1}{\sqrt{2}} = +1\left(x - \frac{1}{\sqrt{2}}\right)$

$\Rightarrow y - \cancel{\frac{1}{\sqrt{2}}} = x - \cancel{\frac{1}{\sqrt{2}}}$

$\boxed{y = x}$ ✓

Exercise 6.3 Chapter ⑥

Q.15 Find the equation of the tangent line to the curve $\boxed{y = x^2 - 2x + 7}$ which is

ⓐ parallel to the line $2x - y + 9 = 0$

ⓑ perpendicular to the line $5y - 15x = 13$.

Ans. $y = x^2 - 2x + 7$

$$\frac{dy}{dx} = 2x - 2 = m_T = \text{Slope of tangent.}$$

$$\boxed{m_T = 2(x-1)}$$

$$\boxed{y = \overset{\text{Slope}}{m}x + c}$$

ⓐ Tangent $\parallel$ $2x - y + 9 = 0$ $\longrightarrow$ $y = 2x + 9$

$$\underset{m_T}{\downarrow} \quad = \quad \underset{m_1 = 2}{\downarrow}$$

$\Rightarrow 2(x-1) = 2$

$\Rightarrow x - 1 = 1$

$\Rightarrow \boxed{x = 2}$

x-coordinate.

$x = 2 \rightarrow$ put $\longrightarrow y = x^2 - 2x + 7$

$y = \cancel{2^2} - \cancel{2\times 2} + 7$

Point $(2, 7)$

Tangent $(y - y_0) = m_T(x - x_0)$

$(y - 7) = 2(x - 2) \Rightarrow y - 7 = 2x - 4$

$\Rightarrow \boxed{y = 2x + 3}$

ⓑ (Tangent) $\perp$ $(5y - 15x = 13)$ $\longrightarrow$ $5y = 15x + 13$

$$\underset{m_T}{\downarrow} \qquad \underset{m_2}{\downarrow} \qquad y = 3x + 13/5$$

$m_T \cdot m_2 = -1$

$$\boxed{m_T = -\frac{1}{m_2}} \Rightarrow \boxed{m_T = -\frac{1}{3}} = 2(x-1)$$

$-\frac{1}{3} = 2(x-1)$

$\Rightarrow -\frac{1}{6} = x-1$

$\Rightarrow 1-\frac{1}{6} = x$

$\Rightarrow \boxed{x = \frac{5}{6}}$

$y = x^2 - 2x + 7$

$y = \left(\frac{5}{6}\right)^2 - 2\left(\frac{5}{6}\right) + 7$

$y = \frac{25}{36} - \frac{5}{3} + \frac{7}{1} = \frac{25-60+252}{36}$

$y = \frac{217}{36}$

$m_T = -\frac{1}{3}$

Tangent $(y - y_0) = m_T (x - x_0)$

$\Rightarrow \left(y - \frac{217}{36}\right) = -\frac{1}{3}\left(x - \frac{5}{6}\right)$

$\Rightarrow \frac{36y - 217}{\cancel{36}\ \cancel{6}\ 2} = -\frac{1}{\cancel{3}}\left(\frac{6x-5}{\cancel{6}}\right)$

$\Rightarrow 36y - 217 = -12x + 10$

$\Rightarrow \boxed{12x + 36y = 227}$ ✓

Q.16 Show that the tangents to the curve $y = 7x^3 + 11$ at the points where $x=2$ and $x=-2$ are parallel.

Ans. $y = 7x^3 + 11$

Slope of tangent $= \frac{dy}{dx} = 21x^2$

Slope of tangent at $(x=2) = 21(2)^2 = 84 = m_1$

slope of tangent at $(x=-2) = 21(-2)^2 = 84 = m_2$

$m_1 = m_2$

$x=2$ Tangent $\parallel$ $x=-2$ Tangent

Q.17 Find the points on the curve $\boxed{y=x^3}$ at which the slope of the tangent is equal to the y-coordinate of the point.

Ans.

Slope of tangent $= m_T = \frac{dy}{dx} = 3x^2$

$\boxed{m_T = 3x^2}$

Let Point be (x, y)

Satisfy

$\boxed{y = x^3}$ — ②

ATQ. $3x^2 = y$ — ①

By eqⁿ ① & ②

$3x^2 = x^3$

$\Rightarrow 0 = x^3 - 3x^2$

$\Rightarrow 0 = x^2(x-3)$

$\boxed{x=0}$ $\boxed{x=3}$

$\boxed{y=x^3}$

$\boxed{x=0} \Rightarrow y = 0^3 \Rightarrow y=0$ $(0,0)$

$\boxed{x=3} \Rightarrow y = 3^3 \Rightarrow \boxed{y=27}$ $(3,27)$

Q.18 For the curve $y = 4x^3 - 2x^5$, find all the points at which the tangent passes through the origin.

last (0,0)

Ans. Let the point be (x_0, y_0) P.

$\boxed{y_0 = 4x_0^3 - 2x_0^5}$ — ①

Tangent at $P(x_0, y_0)$, $\boxed{(y - y_0) = m_T(x - x_0)}$

Slope of tangent $= m_T = \left.\frac{dy}{dx}\right|_{(x_0,y_0)} = \left.(12x^2 - 10x^4)\right|_{(x_0,y_0)}$

$\boxed{m_T = 12x_0^2 - 10x_0^4}$

$y = 4x^3 - 2x^5$

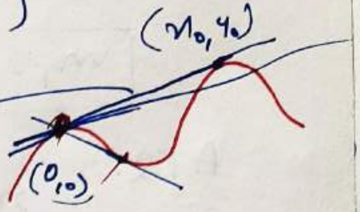
$$\boxed{y_0 = 4x_0^3 - 2x_0^5} \text{ — ①}$$

Equation of tangent $(y - y_0) = m_T(x - x_0)$

$\uparrow$

$(12x_0^2 - 10x_0^4)$

$$\Rightarrow \boxed{(y - y_0) = (12x_0^2 - 10x_0^4)\cdot(x - x_0)}$$



This tangent passes through the origin $(0,0)$

$x = 0$

$y = 0$

$$\Rightarrow (0 - y_0) = (12x_0^2 - 10x_0^4).(0 - x_0)$$

$$\Rightarrow \boxed{y_0 = 12x_0^3 - 10x_0^5} \text{ — ②}$$

$$\boxed{y_0 = 4x_0^3 - 2x_0^5} \text{ — ①}$$

By eqn ① & ②:

$$\Rightarrow 12x_0^3 - 10x_0^5 = 4x_0^3 - 2x_0^5$$

$$\Rightarrow \cancel{8}x_0^5 - \cancel{8}x_0^3 = 0$$

$$\Rightarrow \cancel{8}\ \underbrace{x_0^3}\,\underbrace{(x_0^2 - 1)} = 0$$

$$\Rightarrow \boxed{x_0 = 0},\quad x_0^2 = 1$$

$$\boxed{x_0 = \pm 1}$$

Point

(x_0, y_0)

?

$y_0 = 4x_0^3 - 2x_0^5$

$\boxed{x_0 = 0}$, $\boxed{y_0 = 0}$ $(0,0)$

$\boxed{x_0 = 1}$, $\boxed{y_0 = 2}$ $(1,2)$

$\boxed{x_0 = -1}$, $\boxed{y_0 = -2}$ $(-1,-2)$

$(0,0), (1,2), (-1,-2)$

origin

Q.19 Find the points on the curve $x^2+y^2-2x-3=0$ at which the tangents are parallel to the x-axis.

Ans.

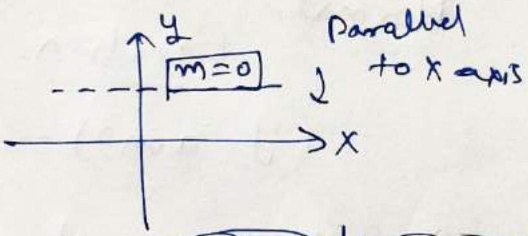
$x^2+y^2-2x-3=0$

by diff. w.r.t. 'x'

$\Rightarrow 2x + 2y\frac{dy}{dx} - 2 = 0$

$\Rightarrow \cancel{2}y\cdot\frac{dy}{dx} = -\cancel{2}x+\cancel{2}^{1}$

$\Rightarrow \boxed{\frac{dy}{dx} = \frac{-x+1}{y}}$ = Slope of tangent —①



tangent || x-axis

$\boxed{m_T = 0}$ —②

$\frac{-x+1}{y} = 0$

$\Rightarrow \boxed{x=1}$ x-coordinate

$x^2+y^2-2x-3=0$

$\Rightarrow 1+y^2-2-3=0$

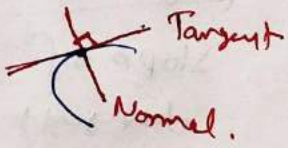
$\Rightarrow y^2=4 \Rightarrow \boxed{y=\pm 2}$ y-coord.

Points

$(1, \pm 2)$

Q.20 Find the equation of the normal at the point (am^2, am^3) for the curve $ay^2=x^3$.

Ans. Normal, $(y-am^3) = m_N\cdot(x-am^2)$



$\boxed{m_N\cdot m_T=-1}$

$\boxed{m_N = -\frac{1}{m_T}}$

$ay^2=x^3$

by diff. w.r.t. 'x'

$2ay\frac{dy}{dx} = 3x^2$

Slope of tangent

$m_T = \frac{dy}{dx} = \frac{3x^2}{2ay}\Big|_{(am^2,\,am^3)} = \frac{3(am^2)^2}{2a(am^3)} = \frac{3\cancel{a^2}m^{\cancel{4}}}{2\cancel{a^2}\cancel{m^3}} = \frac{3m}{2}$

$$m_N = \frac{-1}{\boxed{m_T}} = -\frac{1}{\left(\frac{3m}{2}\right)} = \frac{-2}{3m}$$ ✓

Normal,

Equation.

$$(y - am^3) = \frac{-2}{3m}\left(x - am^2\right)$$

$$\Rightarrow 3my - 3am^4 = -2x + 2am^2$$

$$\Rightarrow 2x + 3my = 3am^4 + 2am^2 = am^2(3m^2+2)$$

$$\boxed{2x + 3my = am^2(3m^2+2)}$$

Q.21 Find the equation of the normal to the ~~curve~~ curve $y = x^3 + 2x + 6$ which are parallel to the line $x + 14y + 4 = 0$

Ans. ~~Norm~~ Normal $\parallel$ $x + 14y + 4 = 0$

$$y = mx + c$$

$$14y = -x - 4$$

$$y = -\frac{x}{14} - \frac{4}{14}$$

$$m_N = m$$

$$m \downarrow -\frac{1}{14}$$

$$\boxed{m_N = -\frac{1}{14}}$$ ✓ —①

$$\boxed{y = x^3 + 2x + 6}$$

Slope of tangent $= \frac{dy}{dx} = 3x^2 + 2 = m_T$

Slope of Normal $= m_N = -\frac{1}{\boxed{m_T}} = -\frac{1}{3x^2+2}$ ✓ —②

By eq^n ① & ②

$$\cancel{-}\frac{1}{3x^2+2} = \cancel{-}\frac{1}{14}$$

$$\Rightarrow 3x^2 + 2 = 14$$

$$\Rightarrow 3x^2 = 12$$

$$\Rightarrow x^2 = 4$$

$$\Rightarrow \boxed{x = \pm 2}$$

$$y = x^3 + 2x + 6$$

$\boxed{x=2}$

$y = 8+4+6$

$y = 18$

$(2, 18)$

$m_N = -\frac{1}{14}$

Eqn. of Normal,

$(y-18) = -\frac{1}{14}(x-2)$

$\Rightarrow 14y - 252 = -x + 2$

$\Rightarrow \boxed{x + 14y - 254 = 0}$

$\boxed{x=-2}$

$y = -8 - 4 + 6$

$y = -6$

$(-2, -6)$

$m_N = -\frac{1}{14}$

Eqn. of Normal,

$(y+6) = -\frac{1}{14}(x+2)$

$\Rightarrow 14y + 84 = -x - 2$

$\Rightarrow \boxed{x + 14y + 86 = 0}$

Q.22 Find the equation of tangent and normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.

Ans. $y^2 = 4ax$

$\Rightarrow \cancel{2}y \cdot \frac{dy}{dx} = \underset{2}{\cancel{4}}a$

$\Rightarrow \boxed{\frac{dy}{dx} = \frac{2a}{y}}$

Slope of tangent at $(at^2, 2at)$ $= m_T = \frac{2a}{2at} = \frac{1}{t}$

Eqn. of tangent,

$\Rightarrow y - y_0 = m_T(x - x_0)$

$\Rightarrow (y - 2at) = \frac{1}{t}(x - at^2)$

$\Rightarrow ty - 2at^2 = x - at^2$

$\Rightarrow \boxed{ty = x + at^2}$

$m_N = -\frac{1}{(m_T)} = -\frac{1}{\left(\frac{1}{t}\right)} = -t$

Eqn. of Normal.

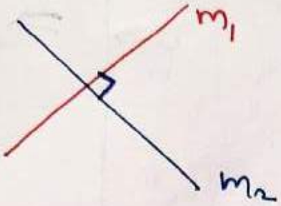
$(y - 2at) = (-t)(x - at^2)$

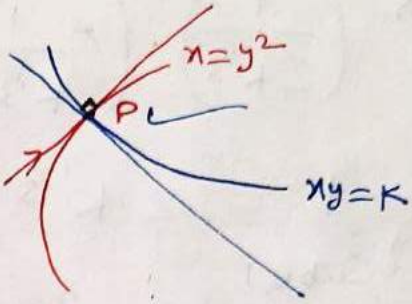
$\Rightarrow \boxed{y = 2at - xt + at^3}$

Exercise 6.3 Chapter-6

Q.23 Prove that the curves $x=y^2$ and $xy=K$ cut at right angles if $8K^2=1$.

$\boxed{m_1 m_2=-1}$





For Point 'p'

$x=y^2$ —①

$xy=K$ —②

Solve.

⇒ $(y^2)y=K$

⇒ $y^3=K$

⇒ $y=K^{1/3}$

$x=(K^{1/3})^2$

$x=K^{2/3}$

Point $P(K^{2/3},K^{1/3})$ ← Point of intersection

$x=y^2$

by diff. w.r.t. 'x'

⇒ $1=2y\cdot\frac{dy}{dx}$

⇒ $\frac{dy}{dx}=\frac{1}{2y}$

$m_1=$ Slope of tangent of $x=y^2$ at 'p' $=\frac{1}{2K^{1/3}}$

$xy=K$

⇒ $y=\frac{K}{x}$

by diff. w.r.t. 'x'

⇒ $\frac{dy}{dx}=-\frac{K}{x^2}$

$m_2=$ Slope of tangent of $xy=K$ at 'p' $=\frac{-K}{(K^{2/3})^2}=\frac{-K^1}{K^{4/3}}$

$=-\frac{1}{K^{1/3}}$

we have $m_1 = \frac{1}{2k^{1/3}}$, $m_2 = -\frac{1}{k^{1/3}}$

Condition of perpendicularity.

$$m_1 m_2 = -1$$

$$\Rightarrow \left(\frac{1}{2k^{1/3}}\right)\cdot\left(\cancel{-}\frac{1}{k^{1/3}}\right) = \cancel{-}1$$

$$\Rightarrow 1 = 2\cdot k^{2/3}$$

(Cube)

$$\Rightarrow 1^3 = \left(2k^{2/3}\right)^3$$

$$\Rightarrow \boxed{1 = 8k^2}$$

✓

Q. 24 Find the equations of the tangent and normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_0, y_0).

Ans.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

by diff. w.r.t 'x'

$$\Rightarrow \frac{\cancel{2}x}{a^2} - \frac{\cancel{2}y}{b^2}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{x}{a^2} = \frac{y}{b^2}\cdot\frac{dy}{dx}$$

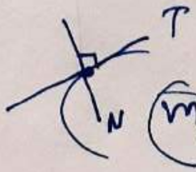
$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{b^2x}{a^2y}}$$

Slope of tangent at (x_0, y_0)

$$m_T = \frac{b^2x_0}{a^2y_0}$$ ✓

Slope of Normal at (x_0, y_0)

$$m_N = -\frac{1}{m_T} = -\frac{a^2y_0}{b^2x_0}$$ ✓



$$m_N \cdot m_T = -1$$

$$m_N = -\frac{1}{m_T}$$

Equation of Tangent

$$(y-y_0) = \frac{b^2 x_0}{a^2 y_0}(x-x_0)$$

$$\Rightarrow \underbrace{\frac{yy_0}{b^2}}\ - \frac{y_0^2}{b^2} = \underbrace{\frac{xx_0}{a^2}}\ - \underbrace{\frac{x_0^2}{a^2}}$$

$$\Rightarrow \underbrace{\frac{x_0^2}{a^2} - \frac{y_0^2}{b^2}}_{1} = \frac{xx_0}{a^2} - \frac{yy_0}{b^2}$$

$$\Rightarrow \boxed{1 = \frac{xx_0}{a^2} - \frac{yy_0}{b^2}}$$

Eqn. of tangent ✓

$$\underbrace{\frac{x^2}{a^2} - \frac{y^2}{b^2}}=1$$

(x_0, y_0) ↑ lies on

$$\boxed{\frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} = 1}$$

Equation of Normal.

$$(y-y_0) = -\frac{a^2 y_0}{b^2 x_0}(x-x_0)$$

$$\Rightarrow \frac{y-y_0}{a^2 y_0} = -\frac{(x-x_0)}{b^2 x_0}$$

$$\Rightarrow \boxed{\frac{y-y_0}{a^2 y_0} + \frac{x-x_0}{b^2 x_0} = 0}$$

Q.25 Find the equation of the tangent to the curve $y = \sqrt{3x-2}$ which is parallel to the line $4x-2y+5=0$.

Ans. Tangent $\parallel$ $4x-2y+5=0$

$$\boxed{m_T = m}$$

$$m_T = 2 \quad —①$$

$$\boxed{y = mx + c}$$

$$2y = 4x + 5$$

$$y = 2x + \frac{5}{2}$$

$$m = 2$$

$$y = \sqrt{3x-2}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{3x-2}} \cdot (3) = \text{Slope of tangent} = m_T \quad —②$$

By eqn ① & ② →

$$2 = \frac{3}{2\sqrt{3x-2}}$$

$$\Rightarrow 4\sqrt{3x-2} = 3$$

$$\Rightarrow 16(3x-2) = 9$$

$$\Rightarrow 3x-2 = \frac{9}{16}$$

$$\Rightarrow 3x = \frac{41}{16}$$

$$\Rightarrow \boxed{x = \frac{41}{48}}$$

$$y = \sqrt{3x-2}$$

$$x = \frac{41}{48} \checkmark$$

$$y = \sqrt{\frac{9}{16}} = \frac{3}{4} \checkmark$$

Point $\left(\frac{41}{48}, \frac{3}{4}\right)$, $m_T = 2$

Equation of tangent.

$$\left(y - \frac{3}{4}\right) = 2\left(x - \frac{41}{48}\right)$$

$$\Rightarrow \frac{4y-3}{\cancel{4}} = \cancel{2}\left(\frac{48x-41}{\cancel{48}}\right) \cancel{12}\ 6$$

$$\Rightarrow 24y - 18 = 48x - 41$$

$$\boxed{23 = 48x - 24y}$$

Q.26 The slope of the normal to the curve $\boxed{y = 2x^2 + 3\sin x}$ at $\boxed{x=0}$ is —

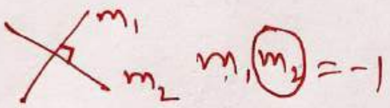
(A) 3 (B) $\frac{1}{3}$ (C) -3 [✓ (D) $-\frac{1}{3}$]

Ans: $y = 2x^2 + 3\sin x$

$\frac{dy}{dx} = \boxed{4x + 3\cos x}$ = Slope of tangent.

Slope of tangent at $(x=0)$ $= 4(0) + 3\cos(0)$
$= 0 + 3 = 3$

Slope of normal at $(x=0)$ $= \frac{-1}{③}$



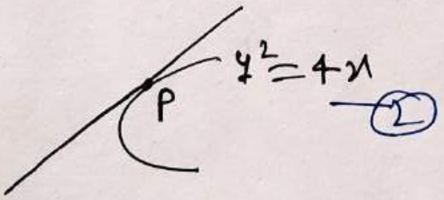
$m_1 \cdot ⓜ_2 = -1$

tangent's slope

Q.27 The line $\boxed{y=x+1}$ is a tangent to the curve $y^2 = 4x$ at the point

[✓ (A) (1,2)] (B) (2,1) (C) (1,−2) (D) (−1,2)

Ans.



Q.9 जैसा

$y = x+1$ —①

By Solving both equations $y = x+1$ & $y^2 = 4x$ for Point P

$\Rightarrow (x+1)^2 = 4x$

$\Rightarrow x^2 + 2x + 1 = 4x$

$\Rightarrow x^2 - 2x + 1 = 0$

$\Rightarrow (x-1)^2 = 0$

$\Rightarrow x - 1 = 0$

$\boxed{x=1}$

$y = x+1$

$y = 1+1$

$\boxed{y=2}$

Point (1,2)

Approximations:

(सन्निकटन)

$\sqrt{100} \downarrow 10$ $\qquad$ $\sqrt{101} \downarrow$

≈ 10.1

≈ 10.01

≈ 10.049 (Calc)

$y = f(x) = \sqrt{x}$

$\sin 30^\circ = \frac{1}{2} = 0.5$

$\sin(30.5)^\circ \approx \frac{1}{2} -$

$\hookrightarrow 0.501$

$\rightarrow y = f(x) = \sin x$

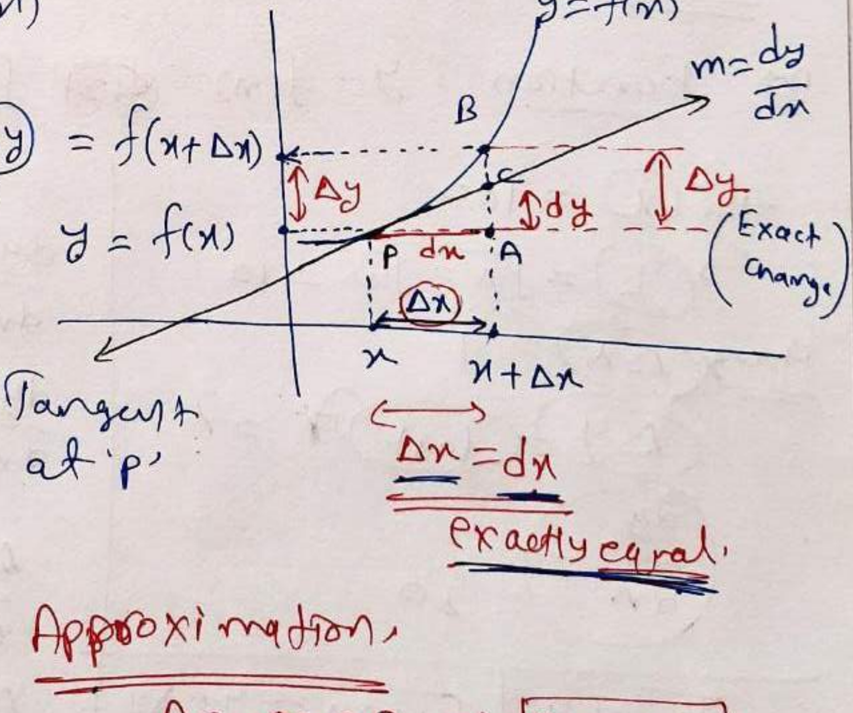
Derivation.

$y = f(x)$

$y + \Delta y = f(x+\Delta x)$

??

$y = f(x)$



input → (x)

↓

no Approximation

Output (y की values)

↓

approximation

$\Delta x = dx$

exactly equal.

Approximation.

$AC \approx AB \Rightarrow \boxed{AC = AB \text{ approx}}$

$\boxed{dy \approx \Delta y}$

$y = f(x)$

$\underset{?}{y+\Delta y} = f(\underline{x+\Delta x})$

$dx = \Delta x$

$dy \approx \Delta y$

$$\frac{dy}{dx} = \frac{\Delta y}{\Delta x} \quad \text{(approximately)}$$

$$\Rightarrow \boxed{\frac{\Delta y}{\Delta x} = \frac{dy}{dx}} \Rightarrow \Delta y = \frac{dy}{dx} \cdot \Delta x$$

Question (Given)

by diff. (Appoximately)

Question: Find approximate value of $\sqrt{101}$.

Ans: Function $y = f(x) = \sqrt{x}$

$\sqrt{100+1}$

$\sqrt{x+\Delta x}$

यहाँ $x = 100$

$y = \sqrt{x} = \sqrt{100} = 10$

change $\Delta x = 1$

Δy = last में = ?

$\frac{dy}{dx} = \frac{1}{20}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{100}} = \frac{1}{2\times 10} = \frac{1}{20}$$

$$\frac{\Delta y}{\Delta x} = \frac{dy}{dx} \quad \text{(approx.)}$$

$$\frac{\Delta y}{1} = \frac{1}{20}$$

$\sqrt{x} = y$

$\sqrt{100} = 10$

$\sqrt{x+\Delta x} = y + \Delta y$

$\sqrt{100+1} = 10 + \frac{1}{20}$

$= 10.05$ ✓

e.g. If the radius of a sphere is measured as 9 cm with an error of 0.03 cm, then find the approximate error in calculating its volume.

Ans.

$$V = \frac{4}{3}\pi r^3$$

$r = 9$ cm

$\Delta r = 0.03$ cm

$\rightarrow V = \times = \frac{4}{3}\pi(9^3) \times$

$\Delta V = ? =$ approx. error in volume

$\frac{dv}{dr} = 4\pi(9^2)$

$$v = \frac{4}{3}\pi \boxed{r^3}$$

$$\frac{dv}{dr} = \frac{4}{\cancel{3}}\pi(\cancel{3}r^2)$$

$$\frac{dv}{dr} = 4\pi r2$$

$$= 4\pi(9)^2$$

Approximation,

$$\boxed{\frac{\Delta v}{\Delta r} = \frac{dv}{dr}}$$

$$\Rightarrow \Delta v = \frac{dv}{dr} \cdot \Delta r$$

$$\Rightarrow \Delta v = 4\pi(9)^2 . (0.03)$$

Approximate error in Volume $\rightarrow$ $\boxed{\Delta v = 9.72\pi \text{ cm3}}$

Approximate volume $= V + \Delta V$

$$= \frac{4}{3}\pi(9)^3 + (9.72\pi)$$

$\boxed{\text{Exercise 6.4}}$ Chapter 6

$\boxed{\text{Approximation}}$

$$y = f(x)$$

$$\underbrace{y + \Delta y}_{?} = f(x + \Delta x)$$

$$\boxed{\frac{\Delta y}{\Delta x} = \frac{dy}{dx}} \star$$

$$\Delta y = \frac{dy}{dx} \cdot \Delta x$$

Exercise 6.4

Q.1 → (ii), (V), (XIV)

Q.2 Q.3 Q.8 ← Questions with same type.

Q.4 Q.5 Q.9 ← Same type.

Q.6 Q.7 ← Same type.

Exercise-6.4 (Approximation)

$\boxed{\text{Q.1}}$

(ii) $\sqrt{49.5}$ function $y = \sqrt{x}$

$x = 49$, $y = \sqrt{x} = \sqrt{49} = 7$

$\sqrt{49.5}$

$49.5 = x + \Delta x$

$\boxed{\Delta x = 0.5}$

$$y + \Delta y = \sqrt{x + \Delta x} = \sqrt{49.5}$$

$$y = \sqrt{x}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{49}} = \frac{1}{2 \times 7} = \frac{1}{14}$$

Approx. → $\boxed{\frac{\Delta y}{\Delta x} = \frac{dy}{dx}}$

$$\Delta y = \frac{dy}{dx} \cdot \Delta x$$

$$\Delta y = \frac{1}{14} \times 0.5 = 0.035$$

$$\sqrt{49.5} = y + \Delta y = 7 + 0.035 = 7.035$$

Hints.

(i) $\sqrt{25.3} \longrightarrow y=\sqrt{x}$ $\qquad 25.3 = x+\Delta x = 25+0.3$

(iii) $\sqrt{0.6} \longrightarrow y=\sqrt{x}$ $\qquad 0.6=0.60=\underbrace{0.64}_{x}-\underbrace{0.04}_{\Delta x}$

(iv) $(0.009)^{1/3} \longrightarrow y=x^{1/3}$ $\qquad 0.009=\underbrace{0.008}_{x}+\underbrace{0.001}_{\Delta x}$

(V) $(0.999)^{\frac{1}{10}}$ $\qquad \boxed{y=x^{\frac{1}{10}}}$

$0.999=\underbrace{1}_{x}-\underbrace{0.001}_{\Delta x}$

$x=1, \quad y=(1)^{1/10}=1$

$x+\Delta x=0.999$

$\underbrace{1 \quad -0.001}$

$\cancel{\Delta x}$

$\Delta x=-0.001$

$y+\Delta y=(0.999)^{1/10}$

$\quad = (x+\Delta x)^{1/10}$

$\boxed{\frac{\Delta y}{\Delta x}=\frac{dy}{dx}}$

$\Delta y=\frac{dy}{dx}\cdot\Delta x$

$\Delta y=\frac{1}{10}\times(-0.001)$

$\boxed{\Delta y=-0.0001}$

$y=x^{\frac{1}{10}}$

$\frac{dy}{dx}=\frac{1}{10}\cdot x^{\frac{1}{10}-1}$

$=\frac{1}{10}\cdot x^{-\frac{9}{10}}$

$=\frac{1}{10.\,x^{9/10}}$

$\frac{dy}{dx}=\frac{1}{10}$

$(0.999)^{\frac{1}{10}}=y+\Delta y$

$=1-0.0001$

$=0.9999$

Hints.

(VI) $(15)^{1/4} \rightarrow y = x^{1/4} \longrightarrow 15 = \underbrace{16}_{x} - \underbrace{1}_{\Delta x}$ $x = 16, \Delta x = -1$

(VII) $(26)^{1/3} \rightarrow y = x^{1/3} \rightarrow 26 = \underbrace{27}_{x} - \underbrace{1}_{\Delta x}$

(VIII) $(255)^{1/4} \rightarrow y = x^{1/4} \rightarrow 255 = \underbrace{256}_{x} - \underbrace{1}_{\Delta x}$

(IX) $(82)^{1/4} \rightarrow y = x^{1/4} \rightarrow 82 = \underbrace{81}_{x} + \underbrace{1}_{\Delta x}$

(X) $(401)^{1/2} \rightarrow y = x^{1/2} \rightarrow 401 = \underline{400} + \underline{1}$

(XI) $(0.0037)^{1/2} \rightarrow y = x^{1/2} \longrightarrow 0.0037 = \overbrace{0.0036}^{x} + \overbrace{0.0001}^{\Delta x}$

(XII) $(26.57)^{1/3} \rightarrow y = x^{1/3} \rightarrow 26.57 = \underbrace{27}_{x} - \underbrace{0.43}_{\Delta x}$

(XIII) $(81.5)^{1/4} \rightarrow y = x^{1/4} \rightarrow 81.5 = \underbrace{81}_{x} + \underbrace{0.5}_{\Delta x}$

(XIV) $(3.968)^{3/2} \rightarrow y = x^{3/2}$,

$3.968 = \underbrace{4}_{x} - \underbrace{0.032}_{\Delta x}$

$y + \Delta y = (3.968)^{3/2}$

$\downarrow$

$y = x^{3/2}$

$y = (4)^{3/2}$

$y = (2)^3$

$y = 8$

$\frac{\Delta y}{\Delta x} = \frac{dy}{dx}$

$\Delta y = \frac{dy}{dx} \cdot \Delta x$

$= 3 \times (-0.032)$

$= (-0.096)$

$y = x^{3/2}$

$\frac{dy}{dx} = \frac{3}{2} \cdot x^{1/2}$

$= \frac{3}{2} \times (4)^{1/2}$

$= \frac{3}{\cancel{2}} \times \cancel{2}$

$= 3$

$(3.968)^{3/2} = y + \Delta y$

$= 8 - 0.096$

$= 7.904$ ✓

Similar Questions 2, 3, ⑧

Q.8 If $f(x) = 3x^2+15x+5$, then the approximate value of $f(3.02)$ is—

Ⓐ 47.66   Ⓑ 57.66   Ⓒ 67.66   [✓ Ⓓ 77.66]

Ans. $y = f(x) = 3x^2+15x+5$ $f(3.02)$

$3.02 = \underbrace{3}_{x} + \underbrace{0.02}_{\Delta x}$

$y = f(3) = 3(3)^2 + 45 + 5$
$= 77$

$$\boxed{y+\Delta y = f(3.02)}$$

$y = 3x^2+15x+5$

$\frac{dy}{dx} = 6x+15$
$= 6(3)+15$
$= 33$

$\frac{\Delta y}{\Delta x} = \frac{dy}{dx} \Rightarrow \Delta y = \frac{dy}{dx} \cdot \Delta x$

$\Delta y = 33 \times 0.02$

$\Delta y = 0.66$

$$\boxed{f(3.02) = y+\Delta y = 77+0.66 = 77.66}$$

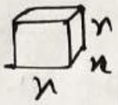
Similar Questions 4, 5, ⑨

Q.9 The approximate change in the volume of a cube of side x meters caused by increasing the side by 3% is—

Ⓐ $0.06\,x^3\ m^3$ Ⓑ $0.6\,x^3\ m^3$

[✓ Ⓒ $0.09\,x^3\ m^3$]   Ⓓ $0.9\,x^3\ m^3$

Ans. $\Delta V = ?$ Volume of a Cube = $V = (\text{Side})^3$



(Side = x)

$V = x^3$

Side increased by 3%.

Now side = $x + \Delta x$

Now side = $x + \underbrace{\Delta x} = x + \underbrace{3\% \text{ of } x}$

$\Delta x = 3\%$ of x

$= \frac{3}{100} \times x$

$= 0.03x$

Approx. $\frac{\Delta V}{\Delta x} = \frac{dV}{dx} \rightarrow$ By diff.

$V = x^3$

$\frac{dV}{dx} = 3x^2$

$\Rightarrow \Delta V = \frac{dV}{dx} \cdot \Delta x$

$\Delta V = (3x^2).(0.03x)$

$\boxed{\Delta V = 0.09x^3}$

approximate change in volume.

$V \pm \Delta V$

Similar Questions ⑥ & 7

Q.6 If the radius of a sphere is measured as 7m with an error 0.02m, then find the approximate error in calculating its volume.

Ans: Sphere → volume $= V = \frac{4}{3}\pi r^3$

$r = 7\,m$

$\Delta r = 0.02\,m$

Approximate volume $= V \pm \Delta V$

Approximate error in volume $= \Delta V$

Approximation

$$\frac{\Delta V}{\Delta r} = \frac{dV}{dr}$$

$$\Rightarrow \Delta V = \frac{dv}{dr} \cdot \Delta r$$

$$\Rightarrow \Delta V = \underbrace{4\pi(49)} \cdot (0.02)$$

$$\Rightarrow \boxed{\Delta V = 3.92\pi \ m3}$$

$V = \frac{4}{3}\pi r^3$

by diff. w.r.t. 'r'

$$\frac{dv}{dr} = \frac{4}{3}\pi(3r^2) = 4\pi r^2$$

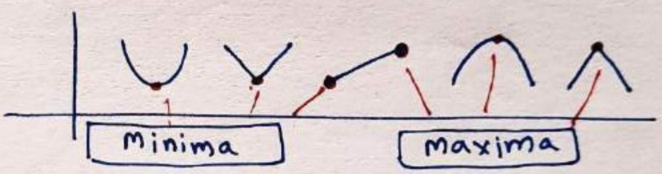
$$\frac{dv}{dr} = \underbrace{4\pi(7)^2}$$

$\Delta V =$ Approximate error in volume $= 3.92\pi \ m3$ ✓

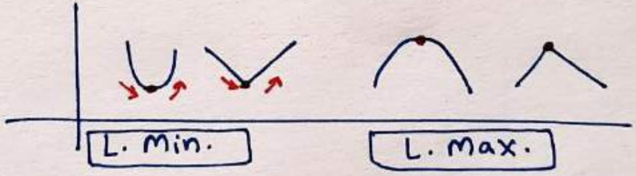
MAXIMA - MINIMA की कहानी

Graphical Meaning of Differentiable, Non-differentiable → Maxima, minima (Extrema) → Turning point (local minima, local maxima) → Critical point → (Ex.) → First order Derivative test to find local extrema → (Ex.) → Second order Derivative test to find local extrema → meaning of $f''(x)$ → concavity of Curve → Point of Inflection → (Ex.) → (Ex.) → Absolute (global) minima/maxim in closed interval → (Ex.) → (Ex.)

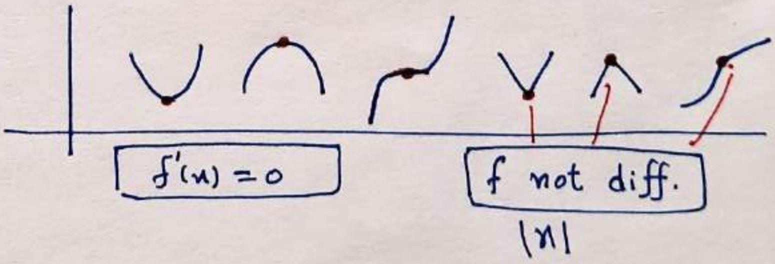
Point of Extrema
(चरम बिंदु)



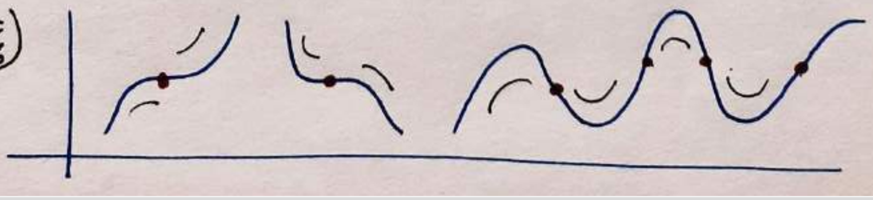
Turning Point
(वर्तन बिंदु)



Critical Point
(क्रांतिक बिन्दु)



Point of Inflection
(नति परिवर्तन बिन्दु)



Maxima and Minima (उच्चतम और निम्नतम) ★

Story

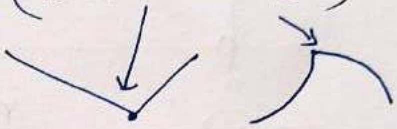
$f(x)$

$x = ?$ कहाँ

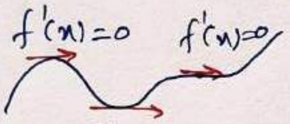
Graphical Meaning of Differentiability.

Differentiable (Smooth)

Non-Differentiable (Sharp Corner)



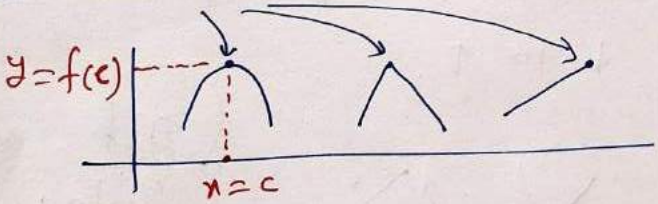
Note.



$f'(x) = 0 = $ Slope

→ Horizontal (flat)

Maxima (maximum) :→



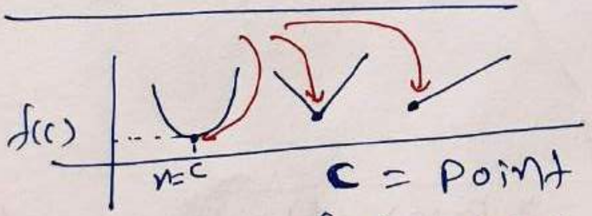
(Peaks)

c = Point of maximum value = Point of maxima

$f(c) =$ maximum value of 'f'

Minima (minimum) :→

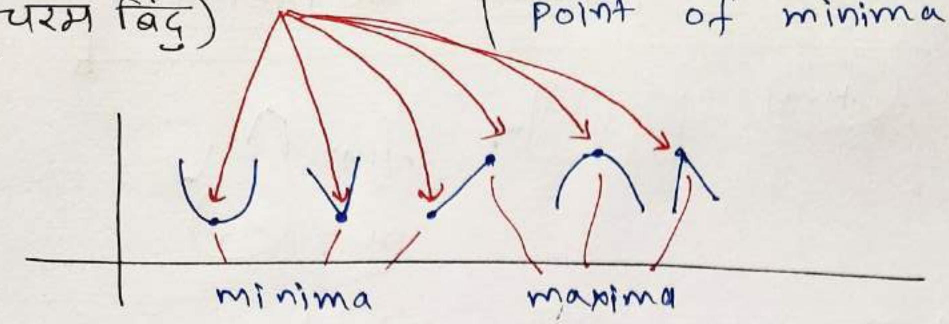
(valley)



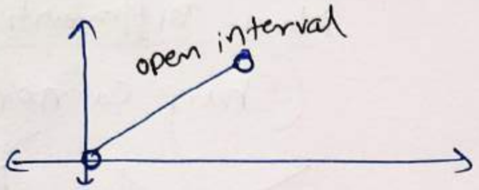
c = Point of minima

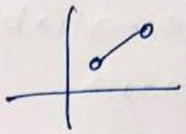
$f(c) =$ value of ~~value~~ minima of 'f'

Point of Extrema (चरम बिंदु) → { Point of maxima, Point of minima }



Note:



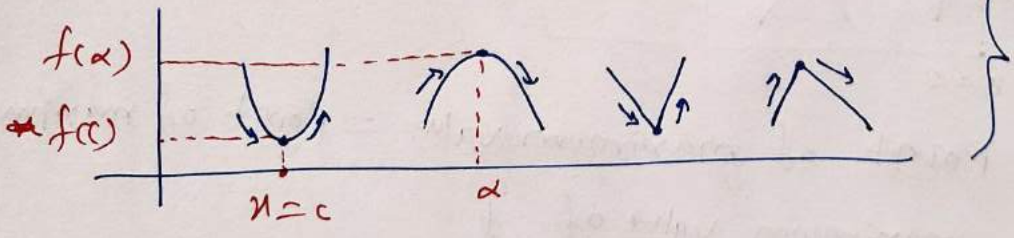


$\frac{\boxed{0.001}}{2} = \frac{\boxed{0.0005}}{2}$ $\boxed{0.00025}$

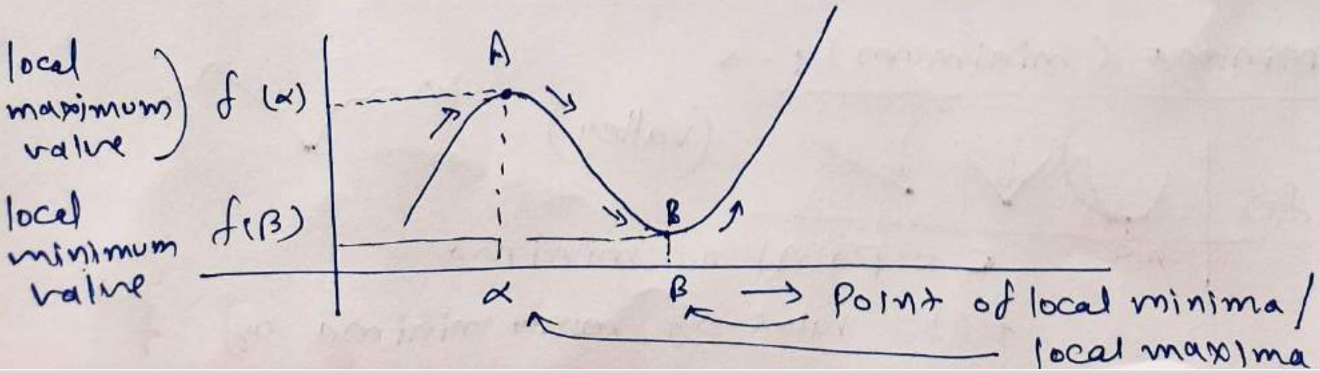
no minima }
no maxima } → no extrema

Turning Point : (वर्तन बिन्दु)

where function changes its nature from ↑ to ↓ or from ↓ to ↑.



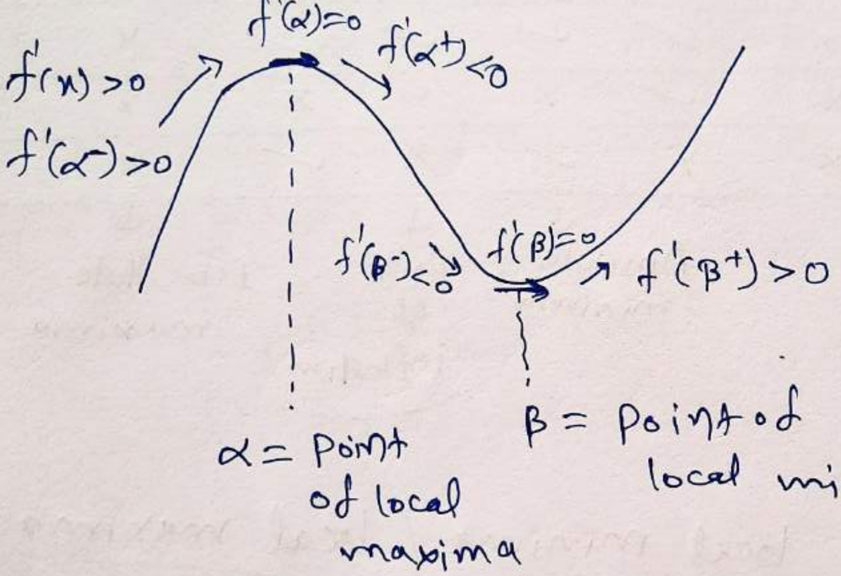
} get Idea

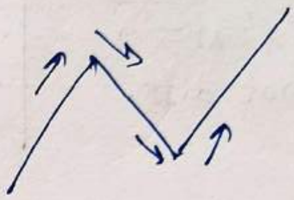


local minimum value
(relative minimum value) } → minimum in very close neighbourhood

local maximum value
(relative maximum value) } → maximum in very close neighbourhood

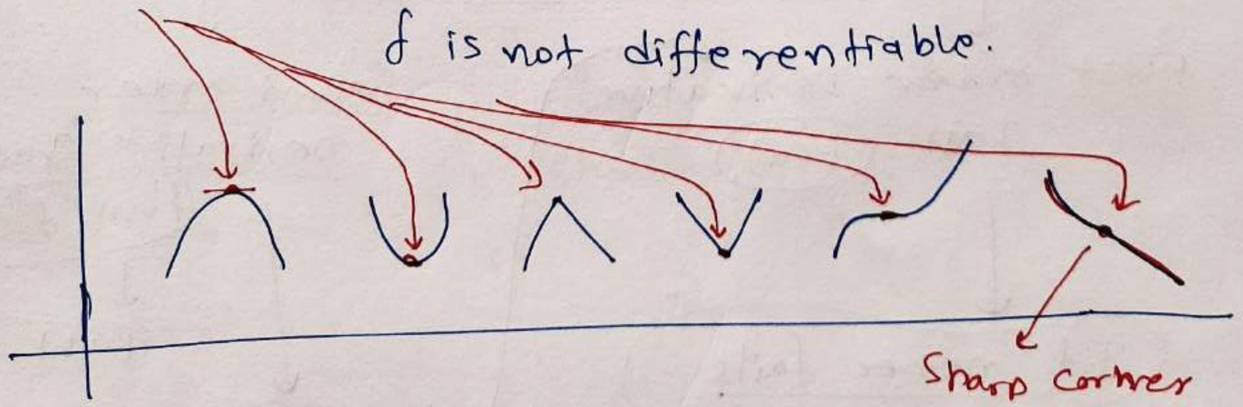
(local minima/ local maxima) → local extrema





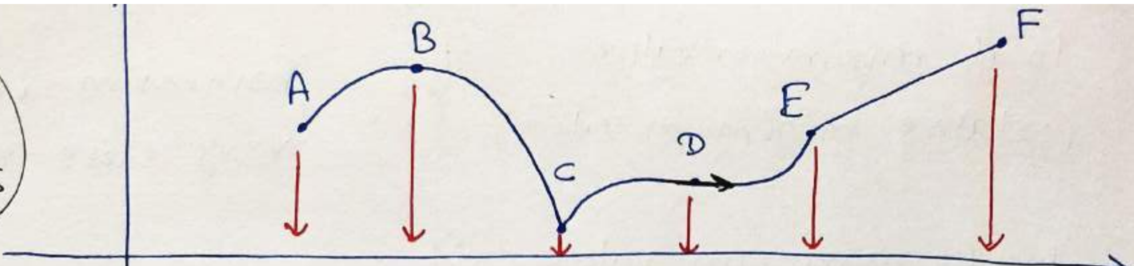
Critical point where $\boxed{f'(x)=0}$ or

f is not differentiable.



e.g.

Table of Tags



	A	B	C	D	E	F
Extrema →	✓	✓	✓	✗	✗	✓
↳ Minima →	✓	✗	✓	✗	✗	✗
↳ maxima →	✗	✓	✗	✗	✗	✓
Turning Point →	✗	✓	✓	✗	✗	✗
↳ Local minima →	✗	✗	✓	✗	✗	✗
↳ Local maxima →	✗	✓	✗	✗	✗	✗
Critical Point →	✗	✓	✓	✓	✓	✗
↳ $f'(x) = 0$ →	✗	✓	✗	✓	✗	✗
↳ f not diff. →	✗	✗	✓	✗	✓	✗
			↓ Absolute minima	↓ Point of inflection		↓ Absolute maxima

How to find local minima / local maxima?

local extrema

- First order Derivative Test $f'(x)$
  - ↓ It never fails
- Second order Derivative Test. $f'(x), f''(x)$
  - ↓ pass
  - ↓ fail → First order Derivative Test

First order Derivative Test to Find Local Extrema

$f'(x) = 0$ ↓ solve $\boxed{x=c}$ ↓

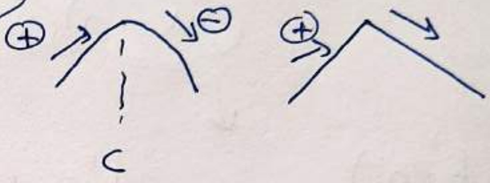
where f not differentiable ↓ $x = c$ ↙ $\boxed{|x|}$

$\boxed{\text{let } x=c \text{ is the critical point}}$

↓

(I) if $f'(c^-) = \oplus$ & $f'(c^+) = \ominus$ then

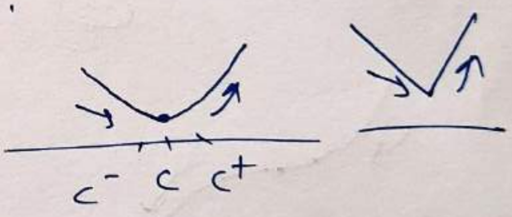
Point of $\boxed{\text{local maxima.}}$



(II) if $f'(c^-) = \ominus$ & $f'(c^+) = \oplus$ then

(पहले) (बाद)

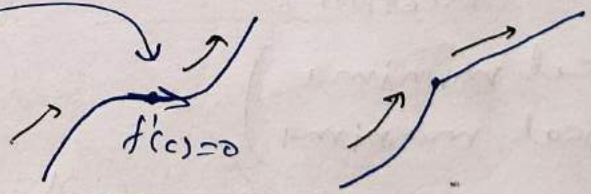
Point of local minima.



(III) If $f'(c^-) = \oplus$ & $f'(c^+) = \oplus$.

then point of inflection.

$\boxed{\begin{array}{l}\text{No local minima}\\ \text{No local maxima}\end{array}}$



$\boxed{\text{Diff.}}$

e.g. Find local minimum value and local maximum value of the function $f(x)=\frac{x^4}{4}-\frac{x^3}{3}+7$.

Ans. 1st order derivative test

$$f(x)=\frac{x^4}{4}-\frac{x^3}{3}+7$$

$\boxed{f'(x)=x^3-x^2}=0$ (Put) (for critical Point)

$\Rightarrow x^2(x-1)=0$

$x=0$, $x=1$

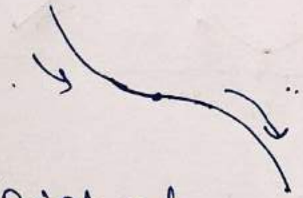
$\boxed{f'(x)=\overset{\oplus}{x^2}(x-1)}$

$f'(0^-)$ ↓ $+ \cdot -$ $=\ominus$

$f'(0^+)$ ↓ $+ \cdot -$ ↓ $\ominus$

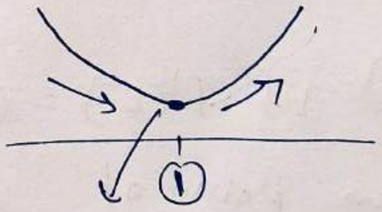
$f'(1^-)$ ↓ $+ \cdot -$ ↓ $\ominus$

$f'(1^+)=$ ↓ $+ \cdot +$ ↓ $\oplus$



(Point of inflection)

(No local minima
No local maxima)



(local minima)

$x=1$ = point of local minima.

No local maxima

local minimum value $=f(1)=\frac{1^4}{4}-\frac{1^3}{3}+7$

$$=\frac{1}{4}-\frac{1}{3}+7$$

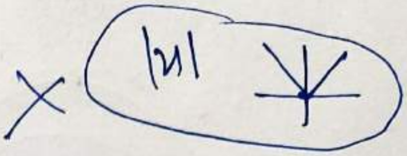
$$=\frac{3-4+84}{12}=\frac{83}{12}$$

Second Order Derivative Test

$f''(x) = \frac{d^2y}{dx^2}$

(only for twice differentiable functions)

×



↓

$f'(x) = 0$ → $x = \alpha$, $x = \beta$, $x = \gamma$

Critical points

↓

$f''(x)$ →

$f''(\alpha) = \oplus ve$ local minima

$f''(\beta) = \ominus ve$ local maxima

$f''(\gamma) = 0$ → 2nd order derivative Test

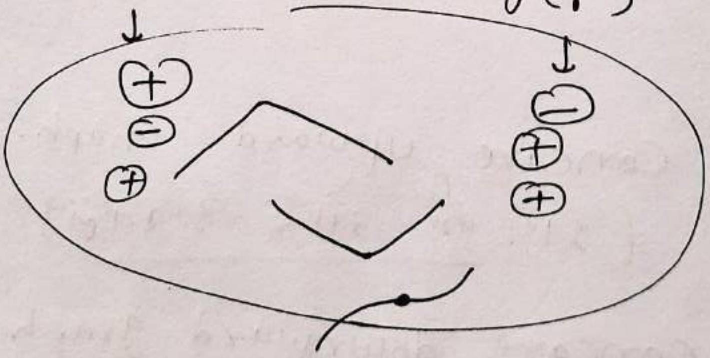
↓

fail

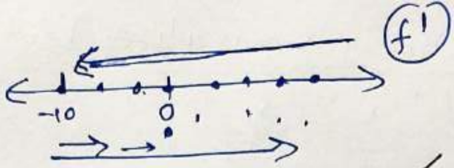
↓

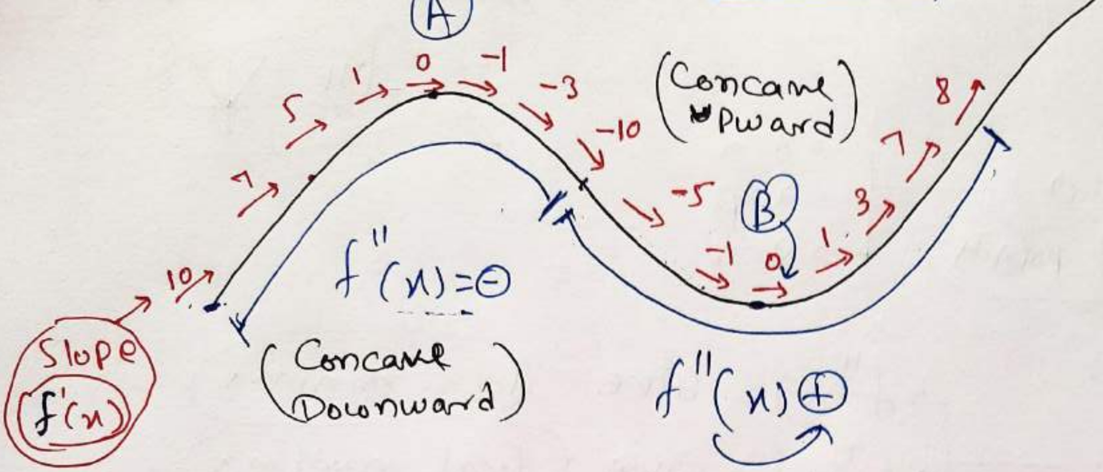
1st order derivative Test

$f'(\gamma-)$ ↓ $f'(\gamma+)$ ↓



Meaning of $f''(x)$





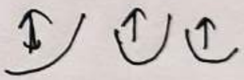
~~Aat~~ at A local maxima $f'(x)=0$

$f''(x) = \ominus$

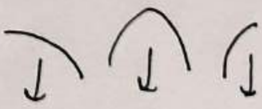
at B local minima $f'(x)=0$

$f''(x) = \oplus$

Note:

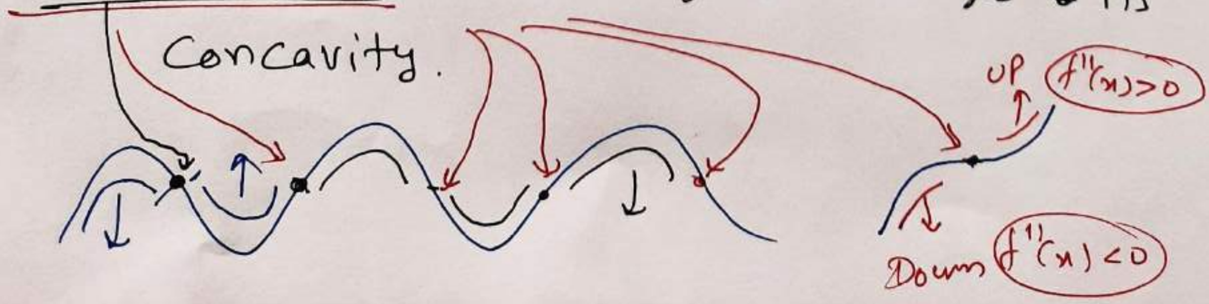
If $f''(x)>0$, Concave upward Graph. 

(उपर की ओर अवतल)

If $f''(x)<0$, concave downward graph 

(नीचे की ओर अवतल)

Point of inflection: where graph changes its concavity.



Q. Find local maximum and local minimum values of the function f given by $f(x) = 3x^4 + 4x^3 - 12x^2 + 12$.

Ans. 2^{nd} order derivative test

$f(x) = 3x^4 + 4x^3 - 12x^2 + 12$

$f'(x) = 12x^3 + 12x^2 - 24x = 0$ (For critical points)

$f'(x) = 12x(x^2 + x - 2) = 0$

$12x(x-1)(x+2) = 0$

$x = 0,\ x = 1,\ x = -2$

$f'(x) = 12x^3 + 12x^2 - 24x$

$f''(x) = 36x^2 + 24x - 24 = 12(3x^2 + 2x - 2)$

$f''(0) = -24 = \ominus ve$ $x = 0$ पर local maxima

$f''(1) = 36 = \oplus ve$ $x = 1$ पर local minima

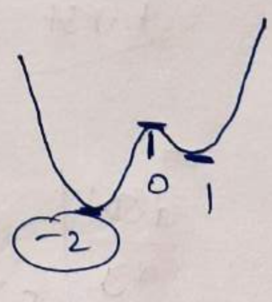
$f''(-2) = 72 = \oplus ve$ $x = -2$ पर local minima.

local maximum value $= f(0) = 12$

local minimum value $= f(1) = 7$

local minimum value $= f(-2) = -20$

$f(x) = 3x^4 + 4x^3 - 12x^2 + 12$ ✓

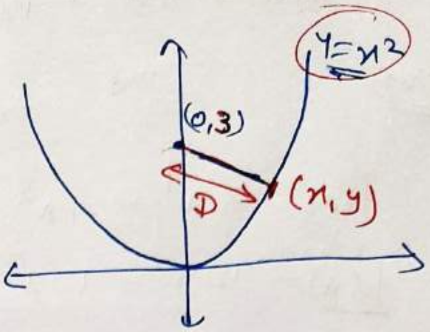


e.g. Find the shortest distance of the point (0, 3) from the parabola $y = x^2$.

↓
Y-axis

Ans.

h



$y = x^2$ — ①

Distance between (0,3) & (x, y) = D = $\sqrt{(x-0)^2+(y-3)^2}$

$\Rightarrow D = \sqrt{\boxed{x^2}+(y-3)^2}$

$\boxed{x^2 = y}$

$\Rightarrow D = \sqrt{\boxed{y+(y-3)^2}}$

by diff. w.r.t. 'y'

$\Rightarrow \frac{d(D)}{dy} = \frac{1.(1+2(y-3).)}{2\sqrt{y+(y-3)^2}}$

$\Rightarrow \frac{d(D)}{dy} = \frac{2y-5}{2\sqrt{y+(y-3)^2}} = 0$ (For Critical Point)

Shortest Distance

$\frac{d(\sqrt{x})}{dx} = \frac{1}{2\sqrt{x}}$

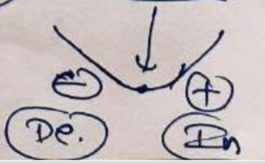
First order Derivative Test.

↓

$\boxed{y = 5/2}$

$\left.\frac{d(D)}{dy}\right|_{5/2^-} = \ominus$ $\left.\frac{d(D)}{dy}\right|_{5/2^+} = \oplus$

minima



$\therefore$ $y = \frac{5}{2}$ is the point of local minima.

$D = \sqrt{y + (y-3)^2}$

$\uparrow$

Distance → minimum ← Shortest Distance

$y = 5/2$ put.

Shortest Distance $= \sqrt{\frac{5}{2} + \left(\frac{5}{2} - 3\right)^2}$

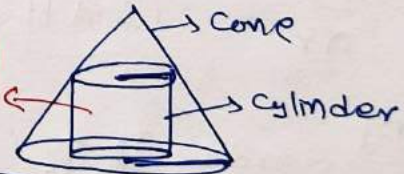
$= \sqrt{\frac{5}{2} + \frac{1}{4}} = \sqrt{\frac{11}{4}} = \frac{\sqrt{11}}{2}$

---

e.g. Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given ~~a~~ cone is half of that of the cone. Constant

Ans. $CSA = 2\pi r h$ (Cylinder)

Greatest CSA



To prove: $\boxed{r_{cylinder} = \frac{1}{2} \cdot r_{cone}}$

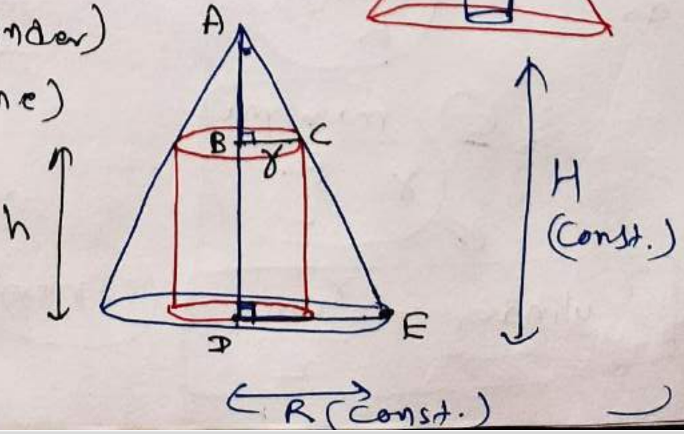
Proof: r, h → variable (cylinder)

R, H → Constant (Cone)

Rule of ~~Sym~~ Similarity (AA)

$\Delta ABC \sim \Delta ADE$

$\frac{AB}{AD} = \frac{BC}{DE} \Rightarrow \boxed{\frac{H-h}{H} = \frac{r}{R}}$



$$\frac{H-\enclose{circle}{h}}{H}=\frac{r}{R}$$

$$\Rightarrow RH-R\enclose{circle}{h}=rH$$

$$\Rightarrow \underline{RH}-r\underline{H}=R\enclose{circle}{h}$$

$$\Rightarrow \frac{H(R-r)}{R}=\enclose{circle}{h}$$

CSA of cylinder $= A = 2\pi r h$

Greatest

Differentiation

$$CSA = \text{'A'} = 2\pi r\enclose{circle}{h}$$
(let)

$$\Rightarrow A = 2\pi r\left(\frac{H(R-r)}{R}\right)$$

$$\Rightarrow A = 2\pi r H - \frac{2\pi r^2 H}{R}$$ $\boxed{\text{variable} = r}$

by diff. w.r.t. 'r'

$$\frac{dA}{dr} = \enclose{circle}{2\pi H} - \frac{2\pi H}{R}(2r) = 0 \text{ (For Critical Points)}$$

$$\Rightarrow 2\pi H\left(1-\frac{2r}{R}\right)=0$$

$$\Rightarrow 1-\frac{2r}{R}=0$$

$$\Rightarrow 1=\frac{2r}{R}$$

$$\Rightarrow \boxed{r=\frac{R}{2}}$$ Critical Point

when

2nd-order D.T.

$$\frac{d^2A}{dr} = 0 - \frac{2\pi H}{R}(2)$$

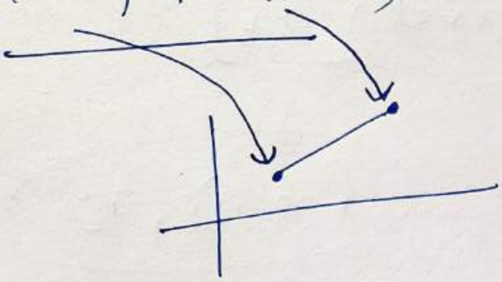
$$= \ominus$$

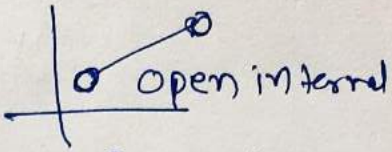
maxima
$$r=\frac{R}{2}$$

$$\boxed{r_{cylinder} = \frac{r_{cone}}{2}}$$

Extrema in the Closed Interval.

(minima / maxima)





no extrema

★ Absolute maxima (Global ~~And~~ maxima, Greatest value)

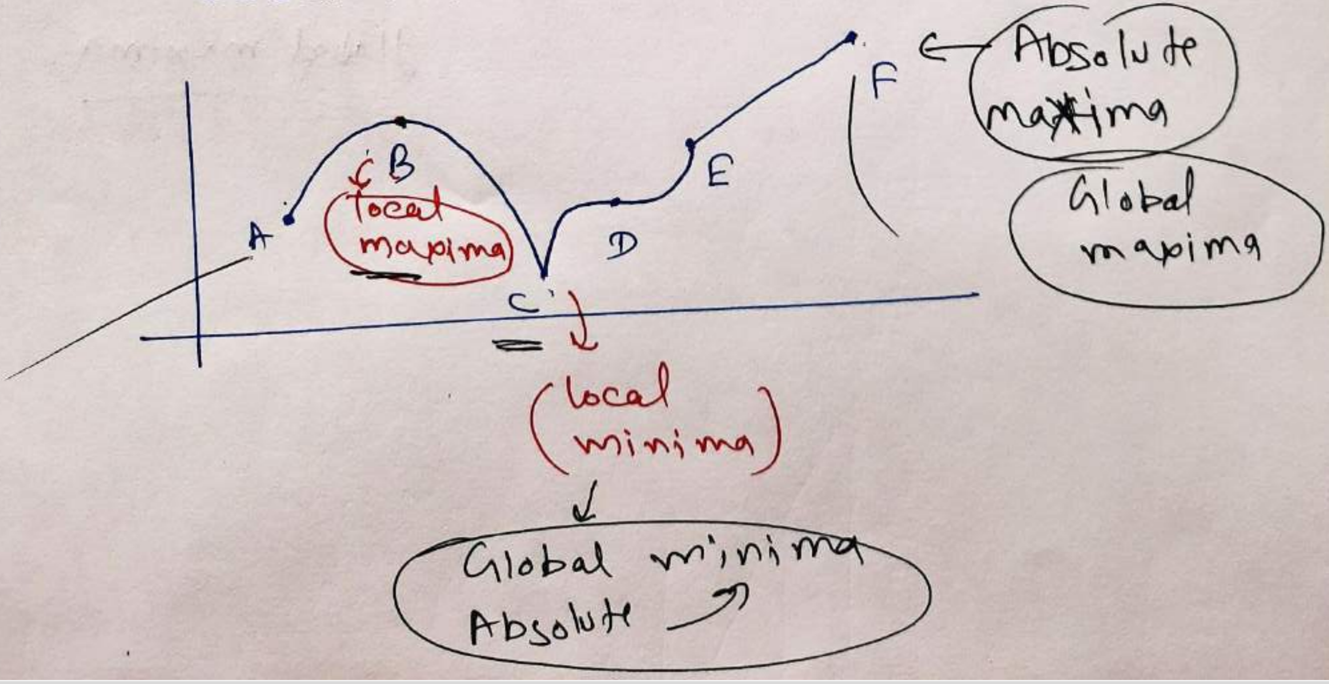
↓

Highest point of the curve in that closed interval.

★ Absolute minima (Global minima, least value)

↓

lowest point of the curve in that closed interval.



How to find absolute (global) minima/maxima ?

$y = f(x)$, interval $[a, b]$

local minima
local maxima

$f'(x) = 0$ ∪ ∩

$x = \alpha$ $x = \beta$

बीच में

किनारे, पर

Find the value of func.

$f(a) =$
$f(b) =$
$f(\alpha) =$
$f(\beta) =$

→ minimum among all these → Global minima

→ maxima among all these → global maxima.

e.g. Find the absolute maximum and minimum of a function f given by $f(x) = 2x^3 - 15x^2 + 36x + 1$ on the interval [1, 5].

Ans.

$$f'(x) = 6x^2 - 30x + 36$$

$$f'(x) = 6(x^2 - 5x + 6)$$

$$f'(x) = 6(x-2)(x-3) = 0$$ (For Critical points)

$$\boxed{x = 2, 3} \in [1, 5]$$

$$\boxed{f(x) = 2x^3 - 15x^2 + 36x + 1}$$

किनारो पर :

$x = 1, \; f(1) = 2 - 15 + 36 + 1 = \boxed{24} \downarrow$

$x = 5, \; f(5) = 2(125) - 15(25) + 36(5) + 1 = \boxed{56} \uparrow$

बीच में

$x = 2, \; f(2) = 16 - 60 + 72 + 1 = 29$

$x = 3, \; f(3) = 54 - 135 + 108 + 1 = 28$

Global (Absolute) maximum value = 56 ($x = 5$)

Absolute minimum value = 24 (at $x = 1$)

e.g Find absolute maximum and minimum values of a function f given by
$f(x) = 12x^{4/3} - 6x^{1/3}$, $x \in [-1, 1]$.

Ans. $f'(x) = 12 \times \frac{4}{3} \cdot x^{\frac{1}{3}} - 6 \times \frac{1}{3} \cdot x^{\frac{1}{3}-1}$ $\boxed{\frac{1}{3} - 1 = -\frac{2}{3}}$

$f'(x) = 16x^{1/3} - \frac{2}{x^{2/3}}$ $\boxed{\frac{1}{3} + \frac{2}{3} = 1}$

$f'(x) = \frac{16x - 2}{x^{2/3}} = \frac{2(8x-1)}{x^{2/3}} = 0$ For Critical Points

$\boxed{f'(x) = 0}$ → Critical point $\boxed{x = \frac{1}{8}}$, $\boxed{x = 0}$ ← Not Differentiable, Not Defined

$\boxed{f'(x) = \infty}$ $\boxed{m = \infty}$

☆

$x \in [-1, 1]$ $\boxed{f(x) = 12x^{4/3} - 6x^{1/3}}$

किनारों पर

$x = -1 \to f(-1) = 18$ → Absolute maxima

$x = 1 \to f(1) = 6$

बीच में

$x = 0 \to f(0) = 0$

$x = \frac{1}{8} \to f\left(\frac{1}{8}\right) = -\frac{9}{4}$ → Absolute minima

Absolute maximum value = 18 (at $x = -1$)

Absolute minimum value = $-\frac{9}{4}$ (at $x = \frac{1}{8}$)

Exercise-6.5 Chapter-6

Q.1 Find the maximum and minimum values, if any, of the following functions given by

(i) $f(x) = (2x-1)^2 + 3$ (ii) $f(x) = 9x^2 + 12x + 2$

(iii) $f(x) = -(x-1)^2 + 10$ (iv) $g(x) = x^3 + 1$

Ans. (i) $\boxed{f(x)} = (2x-1)^2 + 3$, $x \in R$

(Perfect square) ≥ 0

$\Rightarrow (2x-1)^2 \geq 0$

$\Rightarrow \boxed{(2x-1)^2 + 3} \geq 3$

$\Rightarrow f(x) \geq 3$

minimum value of $f(x) = 3$

No maximum value

(ii) $f(x) = 9x^2 + 12x + 2$

$\boxed{a^2 + 2ab + b^2 = (a+b)^2}$

$f(x) = \underbrace{(\overset{\uparrow}{\underset{a}{3x}})^2 + 2.(\underset{a}{3x})(\underset{b}{2}) + 2^2} + 2 - 2^2$

$f(x) = (3x+2)^2 - 2$

we know that $(3x+2)^2 \geq 0$

$\Rightarrow \boxed{(3x+2)^2 - 2} \geq -2$

$f(x) \geq -2$

minimum value of $f(x) = -2$

(no maximum value)

(iii) $f(x) = -\underline{(x-1)}^2 + 10$

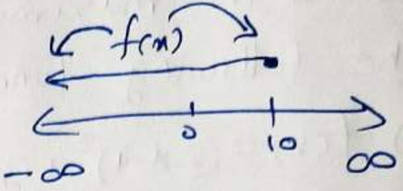
$a > b$

$-a < -b$

$(x-1)^2 \geq 0$

$\Rightarrow \quad -(x-1)^2 \leq 0$

$\Rightarrow \quad \boxed{-(x-1)^2 + 10} \leq 10$



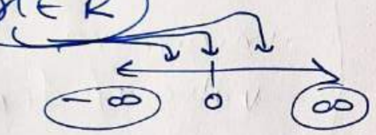
$f(x) \leq 10$

maximum value of f(x) = 10

No minimum value

(iv) $g(x) = x^3 + 1$

$x \in R$



$(-)^3 = -$

$(+)^3 = +$

$(-)^2 = +$

$(+)^2 = +$

$(-)^{odd} = ⊖$

$(+)^{odd} = ⊕$

☆

$x^3 \in R$

$-\infty < x < \infty$

$\Rightarrow \quad -\infty < x^3 < \infty$

$\Rightarrow \quad -\infty < (x^3 + 1) < \infty$

$-\infty < g(x) < \infty$

No maximum value

No minimum value.

Q.2 Find the maximum value and minimum values, if any, of the following functions given by

(i) $f(x) = |x+2| - 1$

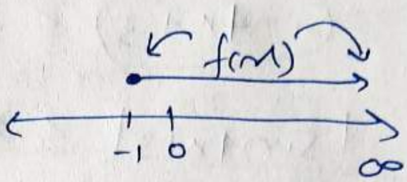
$(x)^2 \geq 0$

$|x| \geq 0$

$|x+2| \geq 0$

$\boxed{|x+2|-1} \geq -1$

$f(x) \geq -1$



minimum value of $f(x) = -1$

no maximum value

(ii) $f(x) = -|x+1| + 3$

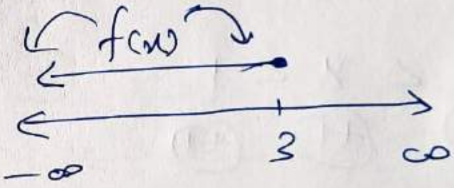
$\boxed{|\ | \geq 0}$

$|x+1| \geq 0$

$\Rightarrow -|x+1| \leq 0$

$\Rightarrow \boxed{-|x+1|+3} \leq 3$

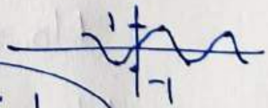
$\Rightarrow f(x) \leq 3$



maximum value of $f(x) = 3$

no minimum value.

(iii) $h(x) = \underline{\sin 2x} + 5$



$\boxed{\begin{array}{c}-1 \leq \sin\theta \leq 1 \\ -1 \leq \cos\theta \leq 1\end{array}}$

we know that $-1 \leq \sin 2x \leq 1$

$+5 \quad +5 \quad +5$

$\Rightarrow 4 \leq \boxed{\sin 2x + 5} \leq 6$

$4 \leq h(x) \leq 6$

minimum value of $h(x) = 4$

maximum value of $h(x) = 6$

(iv) $f(x) = |\underline{\sin 4x} + 3|$

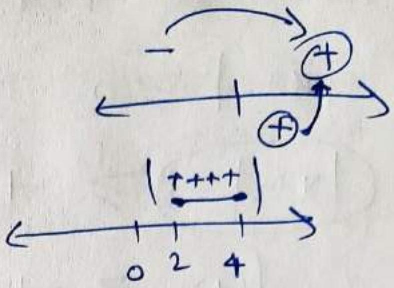
$-1 \le \sin 4x \le 1$
$+3 \qquad +3 \qquad +3$

$2 \le \boxed{\sin 4x + 3} \le 4$

$2 \le |\sin 4x+3| \le 4$

$2 \le f(x) \le 4$

minimum value of $f(x) = 2$
maximum value of $f(x) = 4$



Ⓥ $\underline{h(x)} = \boxed{x+1}, \ \underbrace{x \in (-1,1)}$

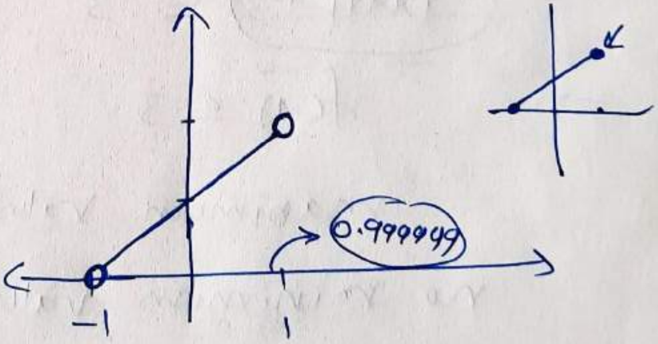
open interval.

$\boxed{y = x+1}$

$-1 < x \le 1$
$\boxed{+1} \quad \boxed{+1} \quad \boxed{+1}$

$0 < \underline{x+1} < 2$

$\boxed{0} < h(x) < \boxed{2}$



No minimum value
No maximum value } ⟵

Q.3 Find the local maxima and local minima, if any, of the following functions. Find also the local maximum and local minimum value.

(i) $f(x) = x^2$

local maxima → Point of local maxima $x=c$

local maximum value → $f(c)$

first order derivative test

second order derivative test → Never fails

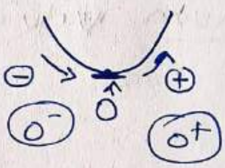
By first order derivative test

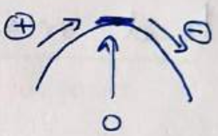
$f(x) = x^2$

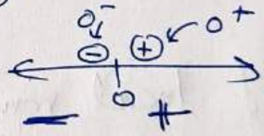
$f'(x) = 2x = 0$ (for critical points)

$\boxed{x=0}$

Slope $= f'(x) = 2x$







$f'(0^-) = 2(0^-)$

$= \ominus$

$f'(0^+) = 2(0^+) = \oplus$

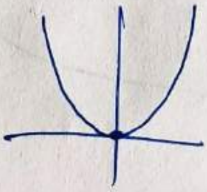
∴ $x=0$ is the Point of local minima.

local minimum value of $f(x) = f(0) = 0^2 = 0$

$\downarrow$

x^2

$y = x^2$



(ii) $g(x) = \underline{x^3 - 3x}$ (second order derivative test)

$g'(x) = 3x^2 - 3 = 0$ (for critical points)

$g'(x) = 3(x^2-1) = 0$

$g'(x) = 3(x+1)(x-1) = 0$

$\boxed{x=-1 \;|\; x=1}$

$g''(x) = 6x \rightarrow$ $g''(-1) = -6 = \ominus$ (local maxima)

$ \rightarrow$ $g''(1) = 6 = \oplus$ (local minima)

Point of local maxima $= x = -1$ $\qquad$ x^3-3x

local maximum value of $g = g(-1) = (-1)^3 - 3(-1)$

$= -1 + 3$

$= 2$

Point of local minima $= x = 1$

local minimum value $= g(1) = 1 - 3 = -2$

(iii) $h(x) = \sin x + \cos x$, $0 < x < \frac{\pi}{2}$

(First order derivative test)

$h'(x) = \cos x - \sin x = 0$ (for critical points)

$x = ?$

$\Rightarrow \cos x = \sin x$

$\Rightarrow 1 = \tan x$

$x = \frac{\pi}{4}$

$f'\left(\frac{\pi}{4}^-\right)$ $\qquad$ $f'\left(\frac{\pi}{4}^+\right)$

$\boxed{h(x) = \sin x + \cos x}$ $\qquad h'(x) = \cos x - \sin x$

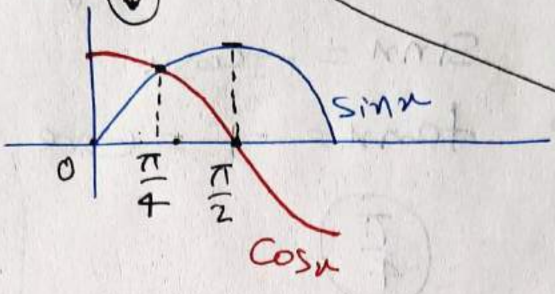
Critical point $\boxed{x = \frac{\pi}{4}} \rightarrow 45^\circ$

$h'\left(\frac{\pi}{4}^-\right) = \underset{\boxed{\uparrow}}{\cos\frac{\pi}{4}^-} - \underset{\boxed{\downarrow}}{\sin\frac{\pi}{4}^-}$

$= \boxed{+}$

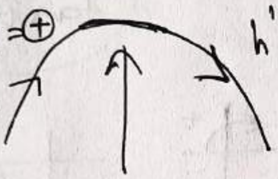
$h'\left(\frac{\pi}{4}^+\right) = \underset{\boxed{\downarrow}}{\cos\left(\frac{\pi}{4}^+\right)} - \underset{\boxed{\uparrow}}{\sin\left(\frac{\pi}{4}^+\right)}$

$= \boxed{-}$



$h'\left(\frac{\pi}{4}^-\right) = \oplus$

increasing



$h'\left(\frac{\pi}{4}^+\right) \ominus$

(Decreasing)

local maxima

$\boxed{x = \frac{\pi}{4}}$

local maximum value $= h\left(\frac{\pi}{4}\right)$

$= \sin\frac{\pi}{4} + \cos\frac{\pi}{4}$

$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}}$

$= \sqrt{2}$

$\frac{2}{\sqrt{2}} = \frac{\cancel{\sqrt{2}} \times \boxed{\sqrt{2}}}{\cancel{\sqrt{2}}}$

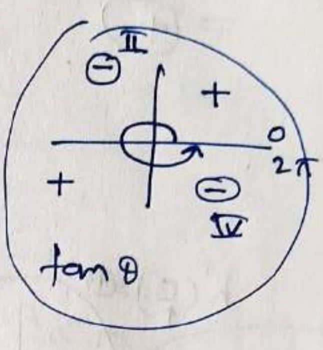
(iv) $f(x) = \sin x - \cos x$, $0 < x < 2\pi$

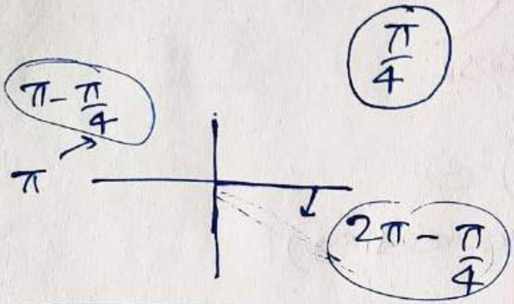
(we will do it by second order derivative test)

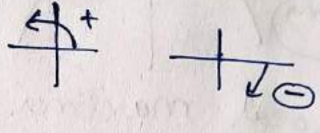
$f'(x) = \cos x + \sin x = 0$ (for critical points)

$\Rightarrow \sin x = -\cos x$

$\Rightarrow \tan x = -1 = \ominus$ve







$\boxed{x = \frac{3\pi}{4}, \; x = \frac{7\pi}{4}}$ Critical points.

$f''(x) = -\sin x + \cos x$

$f''\left(\frac{3\pi}{4}\right) = -\sin\left(\frac{3\pi}{4}\right) + \cos\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}} + \left(-\frac{1}{\sqrt{2}}\right)$

$x = \frac{3\pi}{4}$ (local maxima) $\quad = -\frac{2}{\sqrt{2}} = \ominus$

$f''\left(\frac{7\pi}{4}\right) = -\sin\frac{7\pi}{4} + \cos\frac{7\pi}{4} = +\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \oplus$

$x = \frac{7\pi}{4}$, (local minima) $\quad \boxed{f(x) = \sin x - \cos x}$

local maximum value $= f\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{4} - \cos\frac{3\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$

local minimum value $= f\left(\frac{7\pi}{4}\right) = \sin\frac{7\pi}{4} - \cos\frac{7\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2}$

(I) $f(x) = x^3 - 6x^2 + 9x + 15$

(we will do it by 1st order derivative Test)

$f'(x) = 3x^2 - 12x + 9$

$f'(x) = \underline{3(x^2 - 4x + 3)} = 3(x-1)(x-3) = 0$

$x = 1, 3$ (For critical Points)

$\boxed{x=1}$

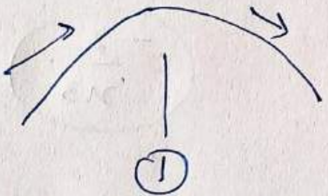
$f'(1^-) = (-).(-) = (+)$

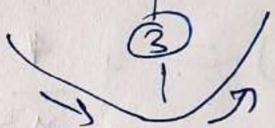
$f'(1^+) = (+ . -) = (-)$

$\boxed{x=3}$

$f'(3^-) = (+)(-) = (-)$

$f'(3^+) = +.+ = (+)$





$\underline{x=1}$ point of local $\underline{\text{maxima}}$

$\underline{x=3}$ point of $\underline{\text{local minima}}$.

$$\boxed{f(x) = x^3 - 6x^2 + 9x + 15}$$

local minimum value $= \underline{f(3)} = 3^3 - 6(3)^2 + 9(3) + 15 = 15$

$\underline{\text{local maximum value}} = \underline{f(1)} = 1^3 - 6(1)^2 + 9(1) + 15 = 19$

(VI) $g(x) = \frac{x}{2} + \frac{2}{x}$, $x > 0$

$\boxed{2^{nd}\text{-order}}$

$g'(x) = \frac{1}{2} - \frac{2}{x^2}$

$g'(x) = \frac{x^2 - 4}{2x^2} = 0$ (for critical points)

$\Rightarrow x^2 - 4 = 0$

$\Rightarrow (x-2)(x+2) = 0$

$\boxed{x = 2, \cancel{-2}}$ $-2 \not> 0$

$g''(x) = 0 - 2\left(-\frac{2}{x^3}\right)$

x^{-2} x^n

$\downarrow$

$-2 . x^{-3}$

$\boxed{\frac{-2}{x^3}}$

$g''(x) = +\frac{4}{x^3}$

$g''(2) = \frac{4}{2^3} = \oplus \longrightarrow x = 2$ Point of local minima)

local minimum value $= g(2) = \frac{2}{2} + \frac{2}{2} = 2$

$\boxed{g(x) = \frac{x}{2} + \frac{2}{x}}$

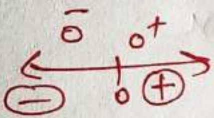
(VII) $g(x) = \frac{1}{(x^2+2)}$ $\boxed{1^{st} \text{ order}}$

$g'(x) = -\frac{1}{(x^2+2)^2} \times (2x)$

$g'(x) = \frac{-2x}{(x^2+2)^2} = 0$ (for critical points)

$\Rightarrow -2x = 0$

$\boxed{x=0}$

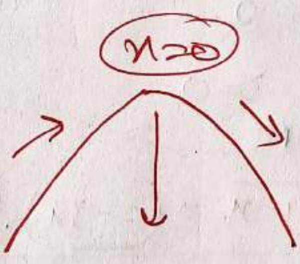


$g'(0^-) = \frac{-(-)}{(+)}$

$= ⊕$

$g'(0^+) = \frac{-(+)}{(+)}$

$= ⊖$



Point of local maxima $x=0$ for $g(x) = \frac{1}{x^2+2}$

local maximum value $= g(0) = \frac{1}{0^2+2} = \frac{1}{2}$

(viii) $f(x) = x.\sqrt{1-x}$, $x > 0$

(2nd order Der)

$$f'(x) = 1.\sqrt{1-x} + x\frac{1}{2\sqrt{1-x}}(0-1)$$

$$f'(x) = \frac{\sqrt{1-x}}{1} - \frac{x}{2\sqrt{1-x}} = \frac{2(1-x)-x}{2\sqrt{1-x}}$$

$$f'(x) = \frac{\boxed{2-3x}}{2\sqrt{1-x}} = 0$$ (for critical points)

$\Rightarrow 2-3x = 0$

$\Rightarrow x = \frac{2}{3} > 0$

$$\left(\frac{u}{v}\right)' = \frac{u'.v - u.v'}{v^2}$$

$$f''(x) = \frac{(-3).2\sqrt{1-x} - (2-3x).\frac{d(2\sqrt{1-x})}{dx}}{4(1-x)}$$

Put $x = \frac{2}{3}$

$$f''\left(\frac{2}{3}\right) = \frac{-3.2\sqrt{1-\frac{2}{3}} - \cancel{0.(\;\;)} \to 0}{4(1-\frac{2}{3})} = \frac{\ominus}{\oplus} = \ominus$$

$x = \frac{2}{3}$ = Point of local maxima.

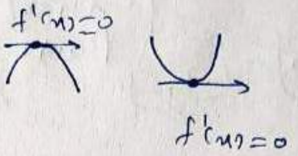
local maximum value $= f(\frac{2}{3}) = x\sqrt{1-x}$

$$= \frac{2}{3}\sqrt{1-\frac{2}{3}} = \frac{2}{3\sqrt{3}}$$

Exercise-6.5 Chapter-6 (maxima & minima)

Q.4 Prove that the following functions do not have maxima or minima :

(i) $f(x) = e^x$



$f'(x) = e^x > 0$

↑

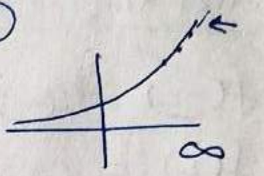
(Slope)

(Increasing)

$e = 2.718\cdots = \oplus$

$(+)^+ = \oplus$

$(+)^- = \oplus$

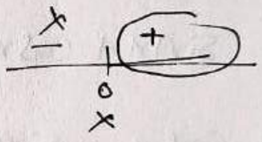


∴ $f(x) = e^x$ does not have any maxima or minima.

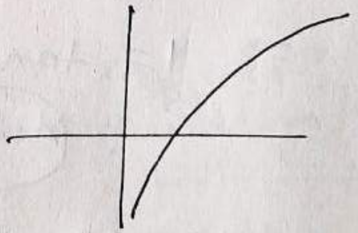
(ii) $g(x) = \log x$

$g'(x) = \frac{1}{x}$

Domain $x > 0$ ✓ $\oplus$



$\frac{1}{x} = \oplus$ ✓



$g'(x) = \frac{1}{x} > 0$ (Increasing)

No maxima, no minima

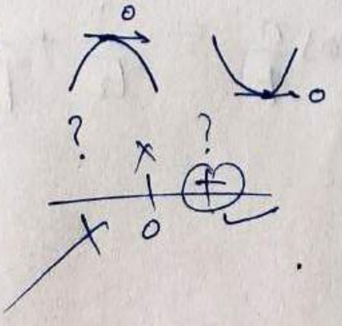
(iii) $h(x) = x^3 + x^2 + x + 1$

$h'(x) = 3x^2 + 2x + 1 \neq 0$

$D = b^2 - 4ac$

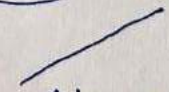
$D = 4 - 4(3)(1)$

$D = -8 < 0$



$h'(x) = 3x^2 + 2x + 1 > 0$

(Slope) > 0 $\oplus$ Increasing



No minima

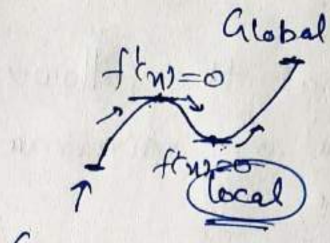
no maxima.

$\boxed{Q.5}$ Find the absolute maximum value and the absolute minimum value of the following function in the given intervals:

(i) $f(x) = x^3$, $x \in [-2, 2]$

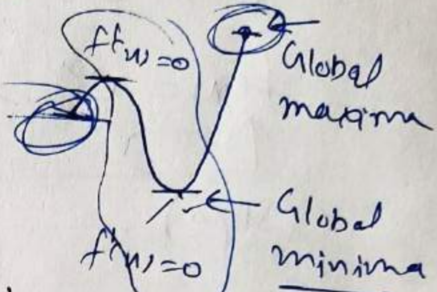
$f'(x) = 3x^2 = 0$ (for Critical Point)

$\Rightarrow \boxed{x=0} \in [-2,2]$



Absolute

global



किनारों पर:

$f(-2) = (-2)^3 = -8$ → Absolute minimum value.

$f(2) = (2)^3 = 8$ → Absolute maximum value

बीच में:

$f(0) = (0)^3 = 0$

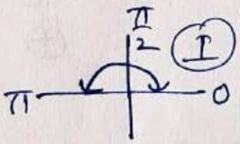
(ii) $f(x) = \sin x + \cos x$, $x \in [0, \pi]$

$f'(x) = \cos x - \sin x = 0$ (for critical points)

$\Rightarrow \cos x = \sin x$

$\Rightarrow 1 = \tan x$

$\oplus$



$\boxed{x = \frac{\pi}{4}}$

$f(0) = \sin 0 + \cos 0 = 0 + 1 = 1$

$f(\pi) = \sin \pi + \cos \pi = 0 + (-1) = -1$ = absolute minimum value.

$f(\frac{\pi}{4}) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$ = Absolute maximum value

(iii) $f(x) = 4x - \frac{1}{2}x^2$, $x \in \left[-2, \frac{9}{2}\right]$

$f'(x) = 4 - x = 0$ (For Critical Points)

$\Rightarrow \boxed{x=4} \in \left[-2, \frac{9}{2}\right]$ ✓ ✓

$f(4) = 4(4) - \frac{1}{2}(4)^2 = 8 \leftarrow$ Absolute maximum value.

$f(-2) = 4(-2) - \frac{1}{2}(-2)^2 = -10$ ✓ $\rightarrow$ Absolute minimum value

$f\left(\frac{9}{2}\right) = 4\left(\frac{9}{2}\right) - \frac{1}{2}\left(\frac{9}{2}\right)^2 = \frac{18}{1} - \frac{81}{8} = \frac{144-81}{8}$

$= \frac{63}{8}$ ✓ $\frac{64}{8}$

(iv) $f(x) = (x-1)^2 + 3$, $x \in [-3, 1]$

$f'(x) = 2(x-1) = 0$ (for Critical points)

$\Rightarrow \boxed{x=1}$

$f(-3) = (-3-1)^2 + 3 = 19 \leftarrow$ Absolute maximum value.

$f(1) = (1-1)^2 + 3 = 3 \leftarrow$ Absolute minimum value.

Q.6 Find the maximum profit that a company can make, if the profit function is given by

$$\boxed{P(x) = 41 - 24x - 18x^2.}$$

Ans.

$P(x) = 41 - 24x - 18x^2$ (Profit function)

$P'(x) = 0 - 24 - 36x = 0 \Leftarrow$ Put

(For critical points)

$\Rightarrow -12(2 + 3x) = 0$

$\Rightarrow \boxed{x = -\frac{2}{3}}$

$P''(x) = 0 - 36$

$P''(x) = -36$

$\boxed{P''\left(-\frac{2}{3}\right) = -36} \rightarrow$ Negative

(local maxima) at $x = -\frac{2}{3}$

local maximum value $= P\left(-\frac{2}{3}\right) = 41 - 24x - 18x^2$

$= 41 - 24\left(-\frac{2}{3}\right) - 18\left(-\frac{2}{3}\right)^2$

$= 41 + 16 - 8$

$= 41 + 8 = 49 \Rightarrow$ maximum profit

$\boxed{Q.7}$ Find both the Absolute maximum value and the minimum value of $3x^4 - 8x^3 + 12x^2 - 48x + 25$ on the interval $[0,3]$.

Ans.

$$f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25 \quad [0,3]$$

$$f'(x) = 12x^3 - 24x^2 + 24x^2 - 48$$

$$f'(x) = 12(\underbrace{x^3 - 2x^2} + \underbrace{2x - 4})$$

$$= 12[x^2(x-2) + 2(x-2)]$$

$$f'(x) = 12[x^2+2].(x-2) = 0$$ (For critical Points)

$x^2 + 2 \neq 0$ $\quad$ $x - 2 = 0$

$x^2 \neq -2$ $\quad$ $\boxed{x = 2} \in [0,3]$

$$\boxed{f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25}$$

$f(0) = 25$ = Absolute maximum value.

$f(2) = \cancel{48} - 64 + \cancel{48} - \cancel{48}(2) + 25 = -39$ → Absolute minimum value.

$f(3) = \underbrace{243 - 216} + \underbrace{108 - 144 + 25}$

$= 27 - 11$

$= 16$

$\boxed{\text{Exercise-6.5}}$ Chapter-6

Minima & Maxima

$\boxed{Q.8}$ At what points in the interval $[0, 2\pi]$, does the function $\sin 2x$ attain its maximum value?

$f'(x)=0$ Absolute maxima

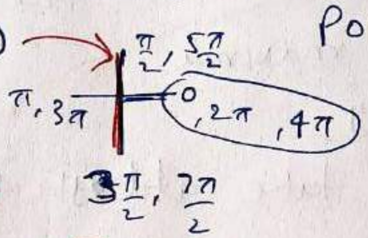
local

$f'(x)=0$

Ans. $f(x) = \sin 2x, \quad x \in [0, 2\pi]$

$f'(x) = 2\cos 2x = 0$ (For critical points)

$\Rightarrow \cos 2x = 0 \longrightarrow$



$x \in [0, 2\pi]$

$2x \in [0, 4\pi]$

$2x = \underbrace{\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}} \in [0, 4\pi]$

$\cos x = \cos\theta$
$x = 2n\pi \pm \theta$

$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ (Critical Point)

$f(x) = \sin 2x$

$f(0) = \sin(2\times 0) = 0$

$f(\frac{\pi}{4}) = \sin(\frac{\pi}{2}) = 1 \longrightarrow$

$f(\frac{3\pi}{4}) = \sin(\frac{3\pi}{2}) = -1$

$f(\frac{5\pi}{4}) = \sin(\frac{5\pi}{2}) = 1 \longrightarrow$

$f(\frac{7\pi}{4}) = \sin(\frac{7\pi}{2}) = -1$

$f(2\pi) = \sin(4\pi) = 0$

maximum values

$x = \frac{\pi}{4}, \frac{5\pi}{4}$

$\boxed{Q.9}$ What is the maximum value of the function $\sin x + \cos x$?

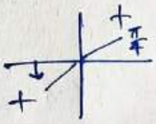
Ans.

$f(x) = \sin x + \cos x$ $\quad x \in R$

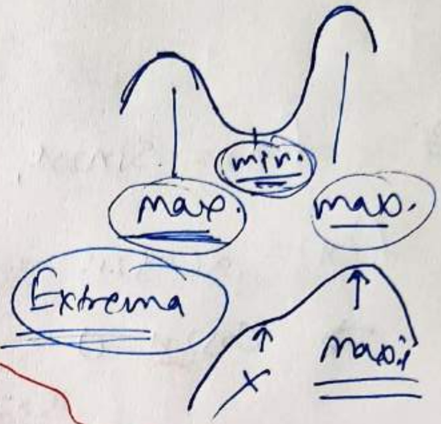
$f'(x) = \cos x - \sin x = 0$ (for critical points)

$\Rightarrow \cos x = \sin x$

$\Rightarrow \boxed{1 = \tan x}$



$\Rightarrow$ $x = \underbrace{\frac{\pi}{4}, \frac{5\pi}{4}}$



Since minima and maxima occurs alternatively, ~~take~~ then we have to take atleast two critical points for complete possibility of maxima.

2nd ~~fe~~ order derivative Test.

$f''(x) = -\sin x - \cos x$, $x = \frac{\pi}{4}, \frac{5\pi}{4}$

$f''\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} - \cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = \ominus$ve.

↑ local maxima

$f''\left(\frac{5\pi}{4}\right) = -\sin\frac{5\pi}{4} - \cos\frac{5\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \oplus$ve

↓ III

local minima.

∴ maximum value of $\sin x + \cos x$ at $\left(x = \frac{\pi}{4}\right)$

↳ $\sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$ ↙

Q.10 Find the maximum value of $2x^3 - 24x + 107$ in the interval $[1,3]$. Find the maximum value of the same function in $[-3,-1]$.

maximum → absolute maximum.

(किनारों पर)

(बीच में)

$f'(x) = 0$ Critical Point.

Ans. $f(x) = 2x^3 - 24x + 107$

$f'(x) = 6x^2 - 24$

$f'(x) = 6(x^2 - 4) = 0$ (for critical points)

$\Rightarrow x^2 = 4$

$\Rightarrow \boxed{x = \pm 2}$

$x = 2, -2$

$[1,3]$ $[-3,-1]$

For maximum value in $[1,3]$

$\boxed{f(x) = 2x^3 - 24x + 107}$

$f(1) = 2 - 24 + 107 = 85$

$f(3) = 54 - 72 + 107 = \textcircled{89}$ ← maximum value in $[1,3]$

$f(2) = 16 - 48 + 107 = 75$

For maximum value in $[-3,-1]$

$f(-3) = \cancel{2} -54 + 72 + 107 = 125$

$f(-1) = -2 + 24 + 107 = 129$

$f(-2) = -16 + 48 + 107 = 139$ ← maximum value in $[-3, -1]$

Q.11 It is given that $x=1$, the function $x^4 - 62x^2 + ax + 9$ attains its maximum value, on the interval $[0,2]$. Find the value of a.

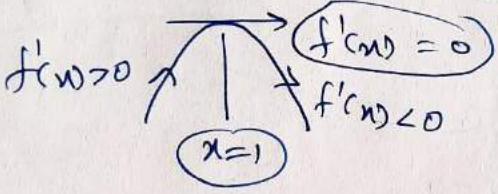
Ans.

$$f(x) = x^4 - 62x^2 + ax + 9$$

$$f'(x) = \underline{4x^3 - 124x^2 + a} = 0 \quad \text{(for critical point)}$$

Satisfy $x=1$ এ $f'(x)=0$

at $x=1$, $f(x)$ gets a maxima.



$\rightarrow \boxed{f'(1)=0}$

$$f'(1)=0$$

$$\Rightarrow 4(1)^3 - 124(1)^2 + a = 0$$

$$\Rightarrow 4 - 124 + a = 0$$

$$\Rightarrow \boxed{a=120}$$

$$\boxed{f(x) = x^4 - 62x^2 + 120x + 9}$$ $x=1$, $x \in [0,2]$

$f(0) = 9$

$f(1) = 1 - 62 + 120 + 9 = \boxed{68}$ maximum

$f(2) = 16 - 248 + 240 + 9 = 17$

$a = 120$

Q.12 Find the maximum and minimum values of $x + \sin 2x$ on $[0, 2\pi]$.

Ans. $f(x) = x + \sin 2x$

Absolute maxima
minima

$f'(x) = 1 + 2\cos 2x = 0$ (For Critical points)

$\Rightarrow \cos 2x = \left(-\frac{1}{2}\right) = \cos\frac{2\pi}{3}$

$x \in [0, 2\pi]$

$2x \in [0, 4\pi]$

Solution of $\cos x = \cos\theta$

$\hookrightarrow \boxed{x = 2n\pi \pm \theta}$

$n \in I$

Solutions

$\cancel{2}x = \cancel{2}n\pi \pm \frac{\cancel{2}\pi}{3}$ $\{\ldots -2, -1, 0, 1, 2, 3, \ldots\}$

$\Rightarrow \boxed{x = n\pi \pm \frac{\pi}{3}}$ $n \in I$

$x \in [0, 2\pi]$

$n = 0$ $x = \pm\frac{\pi}{3}$ → $-\frac{\pi}{3} \notin [0, 2\pi]$
→ $\frac{\pi}{3}$

$n = 1$ $x = \pi \pm \frac{\pi}{3}$ → $\frac{2\pi}{3}$
→ $\frac{4\pi}{3}$

$n = 2$ $x = 2\pi \pm \frac{\pi}{3}$ → $\frac{5\pi}{3}$
→ $\frac{7\pi}{3}$ ✗

Critical points

$x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$, $\underline{x \in [0, 2\pi]}$

$$\boxed{f(x) = x + \sin 2x}$$

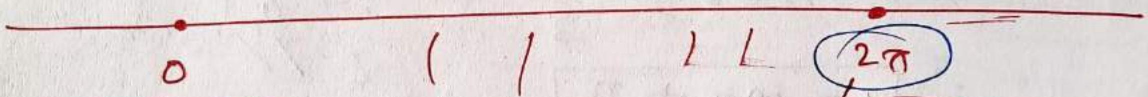
$\underbrace{f(0) = 0 + 0 = \boxed{0}}$, $\underline{f(2\pi) = 2\pi + \overset{\nearrow 0}{\underline{\sin 4\pi}} = \boxed{2\pi}}$

$f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} + \sin\left(\frac{2\pi}{3}\right) = \boxed{\frac{\pi}{3} + \frac{\sqrt{3}}{2}} \times$

$f\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} + \sin\left(\frac{4\pi}{3}\right) = \frac{2\pi}{3} + \sin\left(\pi + \frac{\pi}{3}\right) = \boxed{\overset{\uparrow}{\frac{2\pi}{3}} - \overset{\downarrow}{\frac{\sqrt{3}}{2}}} \times$

$f\left(\frac{4\pi}{3}\right) = \frac{4\pi}{3} + \sin\frac{8\pi}{3} = \frac{4\pi}{3} + \sin\left(2\pi + \frac{2\pi}{3}\right) = \boxed{\frac{4\pi}{3} + \frac{\sqrt{3}}{2}} \times$

$f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} + \sin\left(\frac{10\pi}{3}\right) = \boxed{\frac{5\pi}{3} - \frac{\sqrt{3}}{2}} \times$



6.28...

minimum

at $x = 0$

2π

maximum

at $x = 2\pi$

Exercise 6.5 Chapter ⑥

Q.13 Find two numbers whose sum is 24 and whose product is as large as possible.

$\downarrow$ maximum

maximum | minimum
$\downarrow$
उसी को $f(x)$ (y)
$\downarrow$
Differentiation
$f'(x) = 0$

(1) Variable

Ans.

let x, y be two numbers.

$$x + y = 24$$
$$y = 24 - x$$

let $x \times y \rightarrow$ maximum

$$f(x) = x \times (24 - x) \longrightarrow \text{maximum}$$

1st order Derivative

2nd

$$f(x) = 24x - x^2$$

$$\boxed{f'(x) = 24 - 2x} = 0 \text{ (for Critical points)}$$

$$\Rightarrow 24 = 2x$$

$$\boxed{x = 12}$$

$$f''(x) = 0 - 2 = -2$$

$$f''(12) = -2 = -ve \text{ (local maxima)}$$

x, y
$\downarrow \quad \downarrow$
$12 \quad 12$

$x + y = 24$
$\downarrow \quad \downarrow$
$12 \quad 12$

Q.14 Find two positive numbers x and y such that $x+y=60$ and xy^3 is maximum.

Ans. $x+y=60$

~~$y=(60-x)$~~

$x=60-y$

$xy^3 \longrightarrow$ maximum.

$\Rightarrow \underbrace{(60-y)\cdot y^3}_{f(y)} \longrightarrow$ maximum.

let $f(y)=(60-y).\ y^3$

$f(y)=60y^3-y^4 \longrightarrow$ maxima

$f'(y)=180y^2-4y^3=0$ (for critical points)

$\Rightarrow 4(y^2)[45-y]=0$

$y=0$ ✗, $\boxed{y=45}$ → Check

$x>0$
$y>0$

Second order Derivative Test (SODT)

$f''(y)=360y-12y^2 = 12y(30-y)$

$\boxed{y=45}$

$f''(45)=12\times 45.(-15) = \ominus$ve maxima

$\boxed{y=45}$

$x+y=60$

$\boxed{15}$ $\boxed{45}$

$x.y^3 \longrightarrow$ maxima

Q.15 Find two positive numbers x and y such that their sum is 35 and the product x^2y^5 is a maximum.

Ans.

function,

$x+y=35$

$x=35-y$ — ①

$x^2y^5 \longrightarrow$ maximum

$(35-y)^2.y^5 \longrightarrow$ maximum.

let $f(y)=(35-y)^2\cdot y^5$

by diff. w.r.t. 'y'

$(u.v)'=u'.v+u.v'$

$\Rightarrow f'(y)=-2(35-y).y^5+(35-y)^2.5.y^4$

$\Rightarrow f'(y)=(35-y).y^4.\left[-2y+(35-y).5\right]$

$\Rightarrow f'(y)=(35-y).\,y^4.\,(-7y+35\times5)=0$

(For critical Points)

$y=35$ → $x=0$ ✗

$y=0$ ✗ $y>0$

$\boxed{y=25}$ → maxima

$f'(y)=-7(35-y).y^4(y-25)$

$f'(y)=-7\underbrace{(35.y^4-y^5)}_{u}.\underbrace{(y-25)}_{v}$

$f''(y)=-7(140y^3-5.y^4).(y-25)-7(35.y^4-y^5).1$

$f''(25)=0-7(35\times25^4-25^5)=\ominus$

$y=25, \quad x=10 \qquad (\because x+y=35)$

$\boxed{Q.16}$ Find two positive integers whose sum is 16 and the sum of whose cubes is minimum.

x, y

x^3+y^3

Given $x+y=16$

$y=16-x$

$x^3+(16-x)^3$

$f(x)$

Let $f(x)=x^3+(16-x)^3$

$f(x)=\cancel{x^3}+\cancel{16^3}\,4096-\cancel{x^3}+3(16)(x^2)-3(256)x$

$\Rightarrow f(x)=3\times16\times x^2-3(256)x+4096$

$f'(x)=3\times32\times x-3(256)=0$ (for critical points)

$3(32)(x-8)=0$

$x=8$

$f''(x)=3\times32$

$f''(8)=3\times32=\oplus ve$ minima

$x=8$

$y=8$

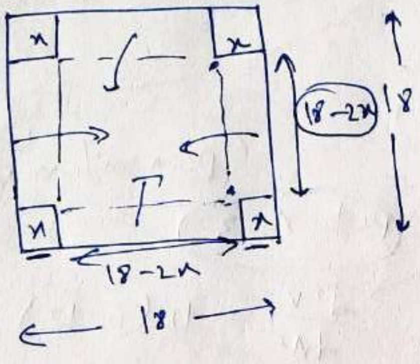
$x+y=16$

8 8

Exercise 6.5 Chapter ⑥

Q.17 A square piece of tin of side 18 cm is to be made into a box without top, by cutting a square from each corner and folding up the flaps to form the box. What should be the side of square to be cut off so that the volume of the box is the maximum possible.

Ans.



minimum / maximum

↓

V, Area, length

↓

1 variable ←

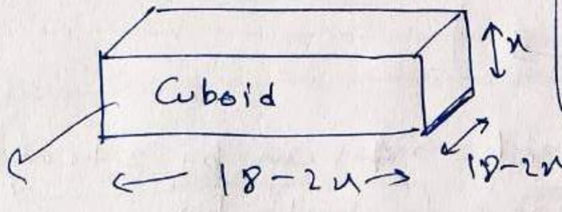
Differentiate $f'(x)=0 < \begin{matrix} x=\alpha \\ x=\beta \end{matrix}$

$f''(x)$

$f''(\alpha)$, $f''(\beta)$

⊕ ⊖

minima maxima



$x = ?$

Box Volume ~~maxi~~ maximum ↑

Volume of Cuboid = l b h

$V = (18-2x) \times (18-2x) \cdot x$

$V = (324 + 4x^2 - 72x)x$

$V = 4x^3 - 72x^2 + 324x$

by differentiating w.r.t 'x'

$\frac{dv}{dx} = 12x^2 - 144x + 324 = 12(x^2 - 12x + 27)$

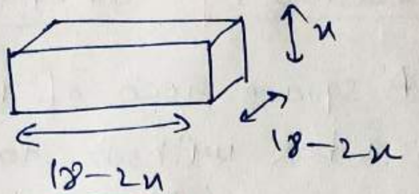
~~$= 12(x^2$~~ $= 12(x-3)(x-9)$

For critical points

put $\frac{dv}{dx}=0$

$\Rightarrow 12(x-3)(x-9)=0$

$x=3,\ \cancel{x=9}$



$x=9$ $18-2x$

$=18-18$ Not

Side $=0$ Possible

~~$f''(x)$~~

$\frac{dv}{dx}=12(x^2-12x+27)$

$\frac{d^2v}{dx^2}=12(2x-12)$

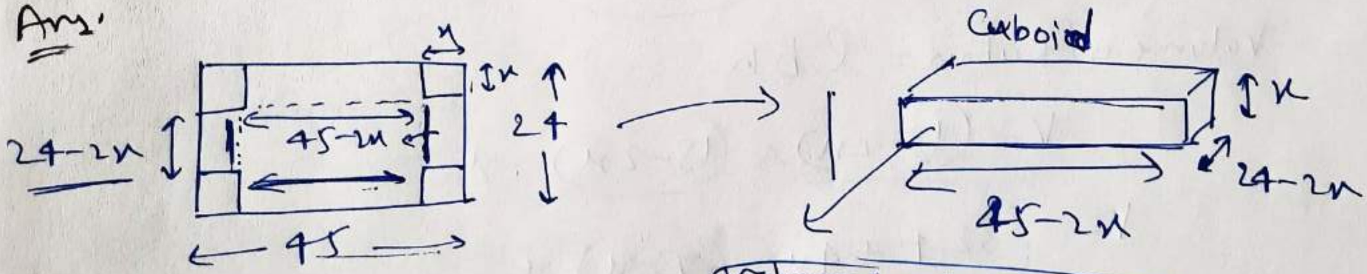
put $x=3$ local point of maxima

$\frac{d^2v}{dx^2}=12(6-12)=\ominus ve$

Side of Square $=x=3$

Q.18 A rectangular sheet → 45 cm by 24 cm.
Cut square from corners to make box.
→ find side? → volume maximum

Ans.



volume → maximum

$v=(45-2x)(24-2x)x$

$v=(1080-138x+4x^2)x$

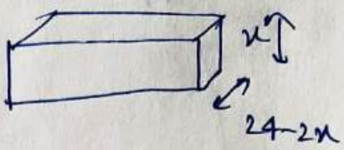
$v=4x^3-138x^2+1080x$

$$\frac{dv}{dx} = 12x^2 - 276x + 1080 = 0$$ (For critical Points)

$$\Rightarrow 12(x^2 - 23x + 90) = 0$$

$$\Rightarrow 12(\underline{x^2 - 18x} - \underline{5x + 90}) = 0$$

$$\Rightarrow 12(x-18)(x-5) = 0$$



$$x = 18, 5$$

✗ ✓

x=18

Side ~~24−36 = −ve~~

$$\frac{d^2v}{dx^2} = 24x - 276$$ (x=5)

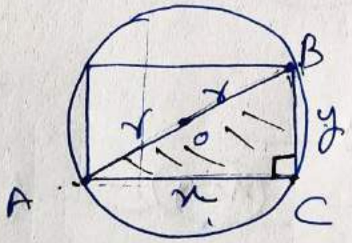
$$= 120 - 276 = \underline{\underline{-ve}}$$

Maxima

$$\boxed{\text{Side of Square} = x = 5}$$

Q.19 Show that of all the rectangles inscribed in a given fixed circle, the square has maximum area.

Ans.



radius = fixed = r (let)
(Constant)

A → max.
A^2 → max.

AB → Rectangle का Diagonal

AB → Circle का Diameter

AB = 2r = Constant

By PGT; $x^2 + y^2 = (2r)^2$

$$y = \sqrt{4r^2 - x^2}$$

Area of rectangle

$$A = x \times y$$

$$A = x \times \sqrt{4r^2 - x^2}$$

Square

$$A^2 = x^2(4r^2 - x^2)$$

$$A^2 = 4r^2x^2 - x^4$$

$A^2 = 4r^2x^2 - x^4$

by diff. w.r.t. 'x'

A → max.

$A^2 \rightarrow$ max.

$\Rightarrow \frac{d(A^2)}{dx} = 8r^2x - 4x^3 = 0$ (For critical points)

$\Rightarrow 4x(2r^2 - x^2) = 0$

$\cancel{x = 0}$, $2r^2 = x^2$

$x = \pm\sqrt{2}r$

$x = +\sqrt{2}r$

$\frac{d^2(A^2)}{dx^2} = 8r^2 - 12x^2$

$= 8r^2 - 12(2r^2)$

$= 8r^2 - 24r^2 = (-)ve$ (maxima)

$x = \sqrt{2}r$ ✓

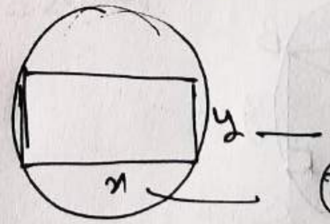
$y = \sqrt{4r^2 - x^2}$

$y = \sqrt{4r^2 - 2r^2}$

$y = \sqrt{2r^2}$

$y = \sqrt{2}r$

$x = y = \sqrt{2}r$

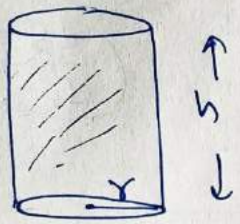


$x = y$

Square → area maximum

Q.20 Show that the right circular cylinder of given surface and maximum volume is such that its height is equal to the diameter of base.

Ans.



Given Surface = total Surface area = Fixed (Constant)

$\boxed{S = 2\pi r h + 2\pi r^2}$ ① S

$r, h \rightarrow$ variable

maximume $\boxed{V = \pi r^2 h}$ ② $\frac{dv}{dr}$

By eqⁿ ①

$$\frac{S - 2\pi r^2}{2\pi r} = h$$

By eqⁿ ②: $V = \pi r^2 h$

$$V = \cancel{\pi} r^{\cancel{2}} \left(\frac{S - 2\pi r^2}{2\cancel{\pi}\cancel{r}} \right)$$

$$V = \frac{Sr}{2} - \pi r^3$$

by diff. w.r.t 'r'

$$\frac{dv}{dr} = \frac{S}{2} - 3\pi r^2 = 0 \text{ (For C.P.)}$$

$$\Rightarrow \frac{S}{2} = 3\pi r^2$$

$$\boxed{S = 6\pi r^2} \rightarrow r = \sqrt{\frac{S}{6\pi}}$$

$$\frac{d^2v}{dr^2} = 0 - 6\pi r$$

$= \ominus$ve (maxima)

$S = 2\pi r h + 2\pi r^2$ ✓

$S = 6\pi r^2$ ✓

$6\pi r^2 = 2\pi r h + 2\pi r^2$

$\Rightarrow \cancel{4}^{2}\cancel{\pi} r^{\cancel{2}} = \cancel{2}\cancel{\pi}\cancel{r} h$

$2r = h$

Diameter → height

$\boxed{Q.21}$ • Of all the closed cylindrical cans, of a given volume of 100 cm^3, find the ~~dimentio~~ dimensions of the can which has the minimum surface area?

Ans.

$$V = 100 = \pi r^2 h \qquad \boxed{\frac{dS}{dr}}$$

$$\boxed{h} = \frac{100}{\pi r^2}$$

$$S = \text{total surface area} = 2\pi r^2 + 2\pi r h$$

$$S = 2\pi r^2 + 2\cancel{\pi}\cancel{r}\left(\frac{100}{\cancel{\pi} r^{\cancel{2}}}\right) = 2\pi r^2 + \frac{200}{r}$$

$$\frac{dS}{dr} = 4\pi r - \frac{200}{r^2} = 0 \quad (\text{for C.P.})$$

$$\Rightarrow 4\pi r = \frac{200}{r^2}$$

$$\Rightarrow 4\pi \boxed{r^3} = 200$$

$$\Rightarrow r^3 = \boxed{\frac{200}{4\pi}} \Rightarrow r = \left(\frac{200}{4\pi}\right)^{1/3}$$

$$\boxed{r = \left(\frac{50}{\pi}\right)^{1/3}}$$

$$\frac{d^2S}{dr^2} = 4\pi + \frac{400}{\boxed{r^3}}$$

$$= 4\pi + \frac{400}{\left(\frac{200}{4\pi}\right)}$$

$$= \oplus ve$$

$$\boxed{\text{minima}}$$

$$h = \frac{100}{\pi r^2} = \frac{100}{\pi \left(\frac{50}{\pi}\right)^{2/3}}$$

$$h = \frac{2\times\overset{50^{1/3}}{\cancel{50}}}{\underset{\pi^{1/3}}{\cancel{\pi}}} \times \frac{\cancel{\pi^{2/3}}}{\cancel{50^{2/3}}} = 2\,\frac{50^{1/3}}{\pi^{1/3}}$$

$$= 2\left(\frac{50}{\pi}\right)^{1/3}$$

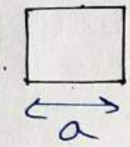
$\boxed{Q.22}$ length of wire = 28 m → Square, Circle → Combined area → minimum

length of two pieces = ?

Ans.

28 m → x , $28-x$

x ↓



$28-x$ ↓

Perimeter of Square $= x$

$\Rightarrow 4a = x$

$$\boxed{a = \frac{x}{4}}$$

Circumference of circle $= 28-x$

$2\pi r = (28-x)$

$$\boxed{r = \frac{28-x}{2\pi}}$$

Combined area $= A$

= area of Square + area of circle.

$= a^2 + \pi r^2$

$$\boxed{A = a^2 + \pi r^2}$$

$$A = \left(\frac{x}{4}\right)^2 + \pi\left(\frac{28-x}{2\pi}\right)^2$$

$$A = \frac{x^2}{16} + \frac{(28-x)^2}{4\pi}$$

↓ minimum

C.P. $0 = \frac{dA}{dx} = \frac{2x}{16} - \frac{2(28-x)}{4\pi}$

$$\Rightarrow \frac{\cancel{2}x}{\cancel{16}\,4} = \frac{\cancel{2}(28-x)}{\cancel{4}\pi}$$

$$\Rightarrow x\pi = 112 - 4x$$

$$\Rightarrow x(\pi+4) = 112$$

$$\Rightarrow x = \frac{112}{\pi+4}$$

$$\frac{d^2A}{dx^2} = \frac{2}{16} + \frac{1}{4\pi}$$

$= (+)$

minima

$28-x$

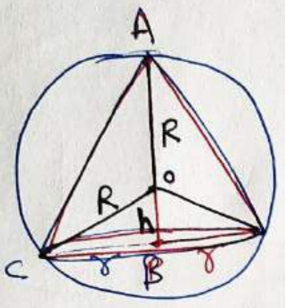
$$= 28 - \frac{112}{\pi+4}$$

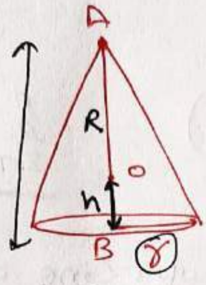
$$= \frac{28\pi + \cancel{112} - \cancel{112}}{\pi+4} = \frac{28\pi}{\pi+4}$$

Exercise 6.5 Chapter 6

Q.23 Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere.

Ans.





Maximum & minimum

Prove → V, A, l

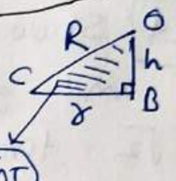
Diff. ← 1 variable

Radius of sphere = R

Radius of cone = r

height of cone = R + h

(Constants = R
variable - r, h)



PGT

$$r^2 + h^2 = R^2$$

Volume of Cone → maxima

$$\boxed{V_c = \frac{1}{3}\pi r^2 (R+h)}$$

$$\boxed{V_c = \frac{1}{3}\pi (R^2 - h^2)(R+h)}$$

$$V_c = \frac{1}{3}\pi (R^3 + R^2 h - R h^2 - h^3)$$

To Prove:

$$\boxed{V_c = \frac{8}{27} \cdot V_s}$$

V_s ↓ $\frac{4}{3}\pi R3$

Diff. w.r.t. 'h'

$$\frac{d(V_c)}{dh} = \frac{\pi}{3}(0 + R^2 - 2Rh - 3h^2) = 0 = \text{(for Critical Points)}$$

$$\frac{d^2(V_c)}{dh^2} = \frac{\pi}{3}(-2R - 6h)$$

↓ $\left(h = \frac{R}{3}\right)$ put

-ve

(maxima)

$\Rightarrow R^2 - 2Rh - 3h^2 = 0$

$\Rightarrow R^2 - 3Rh + Rh - 3h^2 = 0$

$\Rightarrow R(R-3h) + h(R-3h) = 0$

$(R+h).(R-3h) = 0$

X → $\boxed{h = R/3}$

$V_c = \frac{1}{3}\pi(R^2-h^2)(R+h)$

$\left(\text{Put } h=\frac{R}{3}\right)$

$V_c = \frac{1}{3}\pi\left(R^2-\frac{R^2}{9}\right)\left(R+\frac{R}{3}\right)$

$V_c = \frac{1}{3}\pi\left(\frac{8R^2}{9}\right)\cdot\left(\frac{4R}{3}\right)$

$V_c = \frac{8}{27}\left(\frac{4}{3}\pi R^3\right) = \frac{8}{27}\times V_s$

To Prove

$\boxed{V_c = \frac{8}{27}\cdot V_s}$

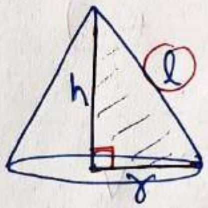
$\downarrow$

$\frac{4}{3}\pi R^3$

Q.24 Show that the right circular cone of least curved surface and given volume has ~~at~~ an altitude equal to $\sqrt{2}$ times the radius of Base.

(height = h)

Ans.



minima

$\boxed{CSA = A = \pi r l}$

Volume = V = given (Fix)

$V = \frac{1}{3}\pi r^2 h$

$h = \frac{3V}{\pi r^2}$

To Prove

$h = \sqrt{2}\, r$

$l^2 = h^2+r^2$

$l = \sqrt{h^2+r^2}$

minima $A = \pi r l$

$A = \pi r\sqrt{h^2+r^2}$

$A = \pi r\sqrt{\frac{9v^2}{\pi^2 r^4}+r^2}$

$A^2 = \pi^2 r^2\left(\frac{9v^2}{\pi^2 r^4}+r^2\right)$

$A^2 = \frac{9v^2}{r^2}+\pi^2 r^4$

$A\downarrow$

$A^2\downarrow$

by Diff. w.r.t. r

$\frac{d(A^2)}{dr} = \frac{-18v^2}{r^3}+4\pi^2 r^3 = 0$

(for C.P.)

$\Rightarrow 4\pi^2 r^3 = \frac{18v^2}{r^3}$

$\Rightarrow \boxed{2\pi^2 r^6 = 9v^2}$

$\Rightarrow \boxed{r^6 = \frac{9v^2}{2\pi^2}}$

$$\frac{d(A^2)}{dr} = -\frac{18v^2}{r^3} + 4\pi^2 r^3$$

again by diff. w.r.t. 'r'

$$\Rightarrow \frac{d^2(A^2)}{dr^2} = +\frac{54v^2}{r^4} + 12\pi^2 r^2$$

put $r^6 = \frac{9v^2}{2\pi^2}$

$= \oplus$ve minima

$A^2 \downarrow$ $A \downarrow$

we got $\rightarrow r^6 = \frac{9(v^2)}{2\pi^2}$ $V = \frac{1}{3}\pi r^2 h$ put

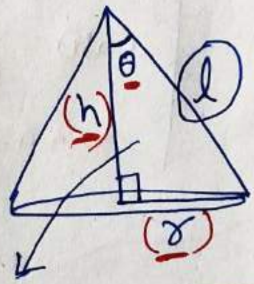
$$\Rightarrow \cancel{r^6} = \frac{\cancel{9}\left(\frac{1}{\cancel{9}} \times \cancel{\pi^2} \times \cancel{r^4} . h^2\right)}{2\cancel{\pi^2}} \quad r^2$$

$$\Rightarrow 2r^2 = h^2$$

$$\Rightarrow \boxed{\sqrt{2} . r = h}$$

Q. 25 Show that the semi vertical angle of the cone of the maximum volume ~~ang~~ and of given slant height is $\tan^{-1}\sqrt{2}$.

Ans



To Prove $\boxed{\theta = \tan^{-1}\sqrt{2}}$

Slant height = l (Constant)

Proof

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi l^2 \sin^2\theta . l\cos\theta$

$V = \frac{1}{3}\pi l^3 . \sin^2\theta . \cos\theta$

$\cos\theta = \frac{h}{l} \Rightarrow \boxed{h = l\cos\theta}$

$\sin\theta = \frac{r}{l} \Rightarrow \boxed{r = l\sin\theta}$

$V = \underline{\frac{1}{3}\pi l^3}\ \underline{\sin^2\theta}\cdot\underline{\cos\theta}$ $\quad (u.v)' = u'.v + u.v'$

by differentiating with respect to 'θ'

$$\Rightarrow \frac{dv}{d\theta} = \frac{1}{3}\pi l^3\left[2.\sin\theta.\underline{\cos\theta.\cos\theta} - \sin^2\theta.\sin\theta\right]$$

$\cos^2\theta = 1 - \sin^2\theta$

$$\frac{dv}{d\theta} = \frac{1}{3}\pi l^3\left[2\sin\theta\,(1-\sin^2\theta) - \sin^3\theta\right]$$

$$\Rightarrow \frac{dv}{d\theta} = \frac{1}{3}\pi l^3\left(2\sin\theta - 3\sin^3\theta\right) = 0$$

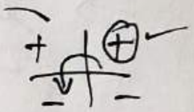
(For Critical Points)

$\Rightarrow \sin\theta(2 - 3\sin^2\theta) = 0$

$\sin\theta = 0$, $\sin^2\theta = \frac{2}{3}$

$\theta = 0, \pi$

$\sin\theta = \pm\sqrt{\frac{2}{3}}$



$\sin\theta = +\sqrt{\frac{2}{3}}$ $\qquad$ $\sin\theta = -\sqrt{\frac{2}{3}}$

S.O.D.T.

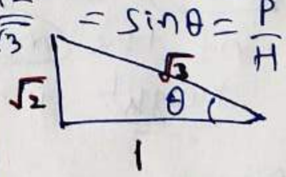
$$\Rightarrow \frac{d^2v}{d\theta^2} = \frac{1}{3}\pi l^3\left(2\cos\theta - 9\sin^2\theta.\cos\theta\right)$$

$$= \frac{1}{3}\pi l^3\left(\frac{2}{\sqrt{3}} - 9.\left(\frac{2}{3}\right)\frac{1}{\sqrt{3}}\right)$$

$\downarrow - \uparrow$

$= \ominus$ve maxima

$\sqrt{\frac{2}{3}} = \sin\theta = \frac{P}{H}$



$\cos\theta = \frac{1}{\sqrt{3}}$

$\tan\theta = \sqrt{2}$

$\theta \rightarrow$ variable $\qquad \sin\theta = \sqrt{\frac{2}{3}}$

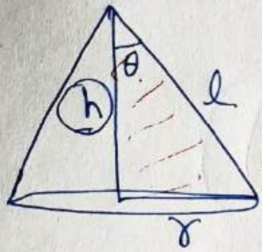
$\tan\theta = \sqrt{2}$

$\boxed{\theta = \tan^{-1}\sqrt{2}}$

Q.26 Show that semi-vertical angle of right circular cone of given surface area and maximum volume is $\sin^{-1}\left(\frac{1}{3}\right)$.

Ans.

To Prove $\boxed{\theta = \sin^{-1}\left(\frac{1}{3}\right)}$



$l = \sqrt{h^2+r^2}$

Surface area $= TSA = A = \pi r l + \pi r^2$

given (Constant)

Variables $= \theta, h, r, l, v$

$h = \sqrt{l^2 - r^2}$

$V = \frac{1}{3}\pi r^2 h$ ← maximum (Diff.)

$V = \frac{1}{3}\pi r^2 \sqrt{l^2 - r^2}$

$A = \pi r l + \pi r^2$

$\left(\frac{A - \pi r^2}{\pi r}\right) = l$

$V = \frac{1}{3}\pi r^2 \sqrt{\left(\frac{A - \pi r^2}{\pi r}\right)^2 - r^2}$

$\Rightarrow V^2 = \frac{1}{9}\pi^2 r^4 \left[\frac{A^2 + \pi^2 r^4 - 2A\pi r^2}{\pi^2 r^2} - r^2\right]$

$\Rightarrow V^2 = \frac{1}{9}\left[A^2 r^2 + \cancel{\pi^2 r^6} - 2A\pi r^4 - \cancel{\pi^2 r^6}\right]$

$\Rightarrow V^2 = \frac{1}{9}\left[A^2 r^2 - 2A\pi r^4\right]$

$V\uparrow = V^2\uparrow$ maximum

$\Rightarrow \frac{d(V^2)}{dr} = \frac{1}{9}\left(2A^2 r - 8A\pi r^3\right) = 0$ (For Critical Point)

$\frac{1}{9}\cdot 2A \cdot r \cdot \left(A - 4\pi r^2\right) = 0$

$\boxed{A = 4\pi r^2}$ $\left(r^2 = \frac{A}{4\pi}\right)$

$$\frac{d(v^2)}{dr} = \frac{1}{9}\left(2A^2r - 8\pi A r^3\right)$$

again by differentiating w.r.t. 'r'

$$\boxed{r^2 = \frac{A}{4\pi}}$$

$$\Rightarrow \frac{d^2(v^2)}{dr^2} = \frac{1}{9}\left(2A^2 - 24\pi A r^2\right)$$

$$= \frac{1}{9}\left(2A^2 - \underset{6}{\cancel{24}}\cancel{\pi}A\left(\frac{A}{\cancel{4\pi}}\right)\right)$$

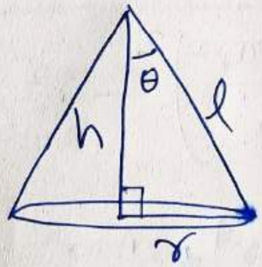
$\downarrow$

$\ominus$ve

$$= \frac{1}{9}\left(2A^2 - 6A^2\right) = \ominus ve$$

$\boxed{\text{maxima}}$

$\boxed{V^2\uparrow} = \boxed{V\uparrow}$



$\boxed{\theta = \sin^{-1}\left(\frac{1}{3}\right)}$ To prove

$$\sin\theta = \frac{r}{l}$$

we got $\boxed{A} = 4\pi r^2$

$\boxed{\text{TSA}}$

$$\Rightarrow \pi r l + \pi r^2 = 4\pi r^2$$

$$\Rightarrow \cancel{\pi}\cancel{r}l = 3\cancel{\pi}r^{\cancel{2}}$$

$$\Rightarrow l = 3r$$

$$\Rightarrow \boxed{\frac{1}{3} = \frac{r}{l}}$$

$$\sin\theta = \frac{r}{l}$$

$$\sin\theta = \frac{1}{3}$$

$$\boxed{\theta = \sin^{-1}\left(\frac{1}{3}\right)}$$

Exercise 6.5 Chapter 6 Q27, Q28, Q29

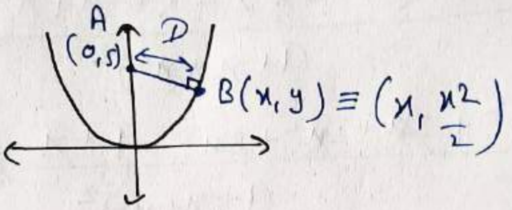
Q.27 The Point on the curve $x^2 = 2y$ which is nearest to the point $(0,5)$ is–

[✓ (A) $(2\sqrt{2}, 4)$] (B) $(2\sqrt{2}, 0)$ (C) $(0,0)$ (D) $(2,2)$

Ans.

$2y = x^2$

$y = \frac{x^2}{2}$



nearest
↓ D
(AB = Minimum)

Distance b/w A & B = $D = \sqrt{(x-0)^2 + \left(\frac{x^2}{2}-5\right)^2}$ $\begin{pmatrix} D\downarrow \\ D^2\downarrow \end{pmatrix}$

$\Rightarrow D^2 = x^2 + \frac{x^4}{4} + 25 - 5x^2$

by Diff.

$\Rightarrow \frac{d(D^2)}{dx} = 2x + x^3 - 10x$

$= x^3 - 8x = 0$ (For Critical Points)

$\Rightarrow x(x^2 - 8) = 0$

$x = 0, \; x^2 = 8 \Rightarrow x = \pm 2\sqrt{2}$

$x = 2\sqrt{2}$ $\quad$ $x = -2\sqrt{2}$ (Not in options)

$\frac{d^2(D^2)}{dx^2} = 3x^2 - 8$

$\boxed{x=0}$ $\frac{d^2(D^2)}{dx^2} = \ominus\text{ve}$ (local maxima)

$\boxed{x = 2\sqrt{2}}$ $\frac{d^2(D^2)}{dx^2} = 16 = \oplus\text{ve}$ (local minima)

at $x = 2\sqrt{2}$ D^2 minima, D minima Shortest.

Point $B\left(x, \frac{x^2}{2}\right)$
$x = 2\sqrt{2}$
$B\left(2\sqrt{2}, \frac{8}{2}\right)$.
$(2\sqrt{2}, 4)$

Q-28 For all real values of x, the minimum value of $\frac{1-x+x^2}{1+x+x^2}$ is — (A) 0 (B) 1 (C) 3 (D) $\frac{1}{3}$

Ans. $f(x)=\frac{1-x+x^2}{1+x+x^2}$ $\left(\frac{u}{v}\right)' = \frac{u'.v-u.v'}{v^2}$

$$f'(x)= \frac{(-1+2x).(1+x+x^2)-(1-x+x^2).(1+2x)}{(1+x+x^2)^2}$$

$$f'(x) = \frac{-1-\cancel{x}-\cancel{x^2}+\cancel{2x}+2x^2+\cancel{2x^3}-1+\cancel{x}-\cancel{x^2}-\cancel{2x}+\cancel{2x^2}-\cancel{2x^3}}{(1+x+x^2)^2}$$

$$f'(x) = \frac{2x^2-2}{(1+x+x^2)^2} = \frac{2(x^2-1)}{(1+x+x^2)^2}$$

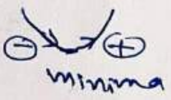
$$f'(x)= \frac{2(x+1)(x-1)}{(1+x+x^2)^2} = 0 \text{ (for critical points)}$$

$\longrightarrow \underbrace{x=-1}, \underbrace{x=1}$

First order Derivative Test

0.9 & पहले — 1.1 (बाद)

1^- $\boxed{x=1}$ 1^+



$\boxed{x=-1}$

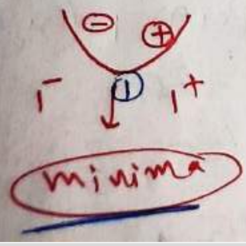
$f'(1^-) = \frac{(+).(-)}{(+)} = ⊖$

Dec.

$f'(1^+) = \frac{+.+}{+} = ⊕$

$f'(-1^-) = \frac{(-).(-)}{(+)} = ⊕$

$f'(-1^+) = \frac{(+).(-)}{(+)} = ⊖$



∴ minimum value at $x=1$

$$f(1) = \frac{1-x+x^2}{1+x+x^2} = \frac{\cancel{1}-\cancel{1}+1}{1+1+1} = \frac{1}{3}$$

[Q.29] The maximum value of $[x(x-1)+1]^{1/3}$, $0 \le x \le 1$ is — (A) $\left(\frac{1}{3}\right)^{1/3}$ (B) $\frac{1}{2}$ [✓ (C) 1] (D) 0

Ans.

maximum, in closed interval $x \in [0,1]$

Absolute maxima
(Global maxima)

$$x^n \downarrow n \cdot x^{n-1}$$

$$f(x) = (x(x-1)+1)^{1/3} = (x^2 - x + 1)^{1/3}$$

$$f'(x) = \frac{1}{3}(x^2-x+1)^{-\frac{2}{3}} \cdot (2x-1)$$

$$f'(x) = \frac{(2x-1)}{3(x^2-x+1)^{2/3}} = 0 \quad \text{(for critical Points)}$$

$$\boxed{x = \frac{1}{2}} \in [0,1]$$

$$f(x) = [x(x-1)+1]^{1/3}$$

$$f(0) = (1)^{1/3} = 1$$

$$f(1) = (1)^{1/3} = 1$$

$$f\left(\tfrac{1}{2}\right) = \left[\frac{1}{2}\left(\frac{1}{2}-1\right)+1\right]^{1/3} = \left[\frac{1}{2} \times \left(-\frac{1}{2}\right)+1\right]^{1/3}$$

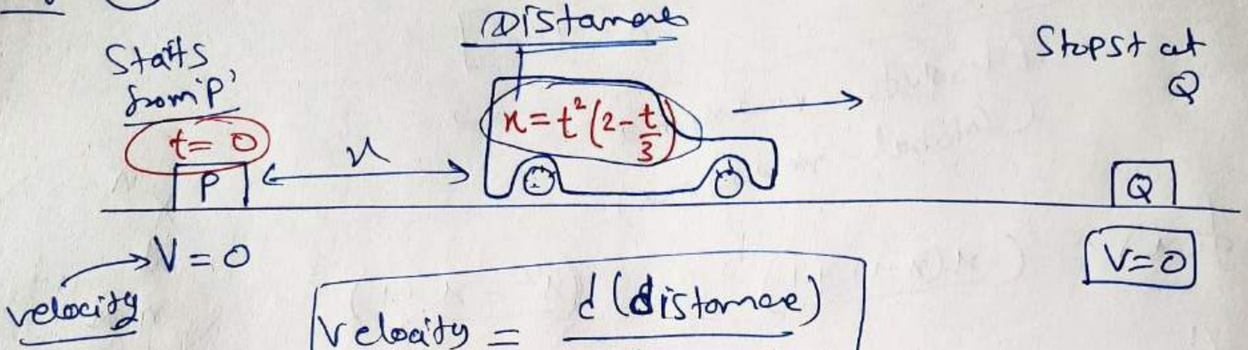
$$= \left(-\frac{1}{4}+1\right)^{1/3} = \left(\frac{3}{4}\right)^{1/3}$$

$$\left(\frac{3}{4}\right)^{\frac{1}{3}} < \underset{\substack{\uparrow \\ \text{maximum}}}{1} = 1$$

Miscellaneous Examples Chapter $\boxed{6}$

$\boxed{42, 43, 44}$

e.g. ㊷



$$\boxed{\text{velocity} = \frac{d(\text{distance})}{d(\text{time})}}$$

$$V = \frac{d(x)}{dt} = \frac{d\left(2t^2 - \frac{t^3}{3}\right)}{dt} = 4t - t^2$$

$$V = t(4-t)$$

at P & Q $v=0$ $\Rightarrow$ $t(4-t)=0$

$t=0$, $\boxed{t=4}$

(P) (Q)

$t = 4\,\text{sec.}$

$P \longmapsto Q$

$\longleftrightarrow$

$$x = t^2\left(2 - \frac{t}{3}\right)$$

$\boxed{t=4}$

$$x = 4^2\left(2 - \frac{4}{3}\right)$$

$$PQ = x = 16\left(\frac{2}{3}\right) = \frac{32}{3}\,m$$

$\boxed{\text{e.g. } (43)}$ water tank ↓ Semivertical angle

water → $\boxed{\text{inverted cone}}$ ▽ $\alpha = \tan^{-1}(0.5)$

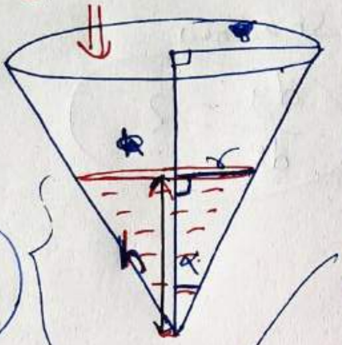
5 m^3/hour

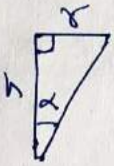
→ $\frac{dv}{dt}$

water level rising rate = ? when depth of water = (4m) = h

solution. $\frac{dv}{dt} = 5$

⇓





$\frac{dh}{dt} = ?$

Rate of change

- D → Diagram
- D → Derivative (✓/?)
- F → Formula
- De → Differentiation
- C → Constants.

$\alpha = \tan^{-1}(0.5)$

$\tan\alpha = \frac{1}{2} = \frac{\text{Perp.}}{\text{Base}} = \frac{r}{h}$

$\frac{1}{2} = \frac{r}{h}$

$h = 2r$

$r = \frac{h}{2}$

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi r^2 (2r)$$

$$V = \frac{2}{3}\pi r^3$$

$$V = \frac{2}{3}\pi\left(\frac{h}{2}\right)^3 = \frac{\pi}{12}h^3$$

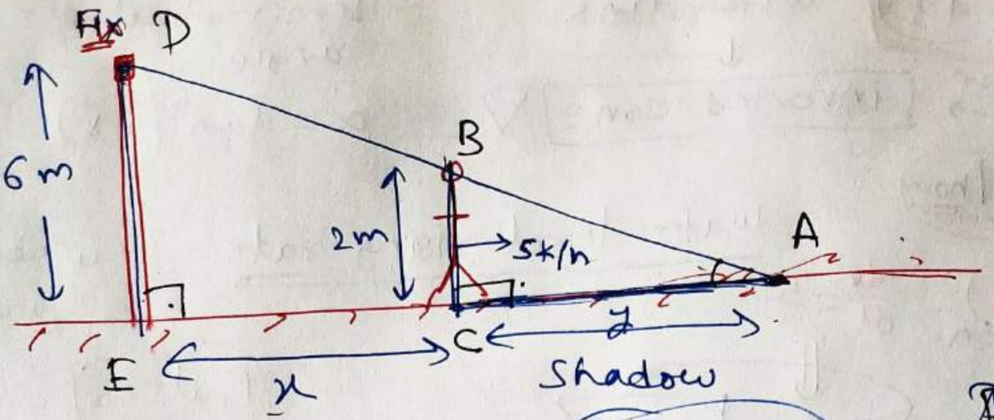
by Diff.

$$\frac{dv}{dt} = \frac{\pi}{12} \times 3h^2 \times \frac{dh}{dt}$$

$$\Rightarrow 5 = \frac{\pi}{12} \times 3(4)^2 \times \frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{5 \times \cancel{12} \times 7}{22 \times \cancel{8} \times \cancel{4} \times 4} = \frac{35}{88} \text{ m/h}$$

e.g. (44)



$\frac{dx}{dt}=5$ $\qquad$ $\frac{dy}{dt}=?$

D✓
D✓
F✓
D✓
C

(AA)

ΔACB ~ ΔAED

$\Rightarrow \frac{AC}{AE}=\frac{CB}{ED}$

$\Rightarrow \frac{y}{x+y}=\frac{\cancel{2}^{1}}{\cancel{6}_{3}}$

$\Rightarrow 3y = x+y$

$\Rightarrow \boxed{2y=x}$

$\boxed{2y=x}$

$\downarrow$

$\Rightarrow 2\frac{dy}{dt}=\frac{dx}{dt} \rightarrow 5$

$\Rightarrow \boxed{\frac{dy}{dt}=\frac{5}{2}\ k/h}$

$\downarrow$

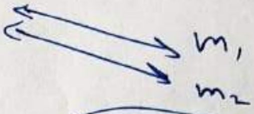
Shadow Rate of Change

Class 12 Maths Application of Derivatives

Miscellaneous Examples Chapter ⑥

(46, 48, 51)

Example: 46 Find the equation of tangents to the curve $\boxed{y=\cos(x+y)}$, $-2\pi \le x \le 2\pi$ that are parallel to the line $x+2y=0$.



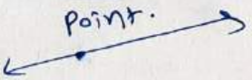
$\boxed{m_1 = m_2}$

$\boxed{y = mx+c}$

$$x+2y=0$$
$$2y=-x$$
$$y=-\frac{x}{2}$$
$$\boxed{m=\frac{-1}{2}}$$

Solution:

$$\boxed{m_T=-\frac{1}{2}} = \boxed{\frac{dy}{dx}}$$



Curve $y=\cos(x+y)$

diff.

$$\Rightarrow \frac{dy}{dx} = -\sin(x+y)\times\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx} = -\sin(x+y) - \sin(x+y)\cdot\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}\left(1+\sin(x+y)\right) = -\sin(x+y)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\sin(x+y)}{1+\sin(x+y)} = \text{Slopp of tangent} \qquad \left(m_T=-\frac{1}{2}\right)$$

equate

$$\Rightarrow \frac{\cancel{-}\sin(x+y)}{1+\sin(x+y)} = \cancel{-}\frac{1}{2}$$

$$\Rightarrow 2\sin(x+y) = 1+\sin(x+y)$$

$$\Rightarrow \boxed{\sin(x+y)=1} \rightarrow ①$$

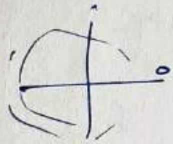
$$\cos(x+y) = \sqrt{1-\sin^2(x+y)} = \sqrt{1-1^2} = 0$$

$\cos(x+y) = 0$

Curve $y = \cos(x+y)$

$\Rightarrow \boxed{y=0}$

Point का y-coordinate

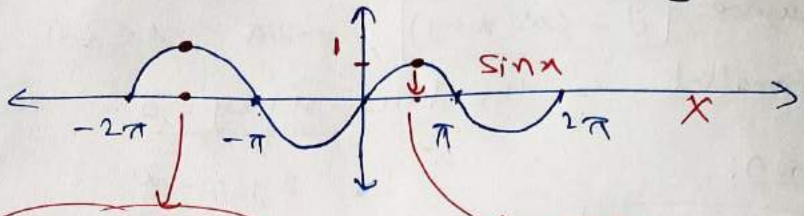


$\sin(x+y) = 1$

Put $y=0$

$\sin(x) = 1$ $x = ?$

$x \in [2\pi, 2\pi]$



$x = -\frac{3\pi}{2}$, $x = \frac{\pi}{2}$

Points $\left(\frac{3\pi}{2}, 0\right)$ $\left(\frac{\pi}{2}, 0\right)$

$m_T = -\frac{1}{2}$

$\downarrow$

$(y - y_0) = m_T(x - x_0)$

$\downarrow$

$$(y-0) = -\frac{1}{2}\left(x + \frac{3\pi}{2}\right)$$

$m_T = -\frac{1}{2}$

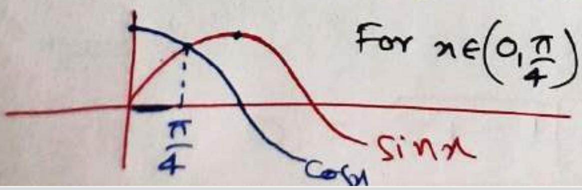
$\longrightarrow$

$$\boxed{(y-0) = -\frac{1}{2}\left(x - \frac{\pi}{2}\right)}$$

Example ㊽ Show that the function f given by $f(x) = \tan^{-1}(\sin x + \cos x)$, is always an increasing function in $\left(0, \frac{\pi}{4}\right)$.

Ans. $f(x) = \tan^{-1}(\sin x + \cos x)$ $x \in \left(0, \frac{\pi}{4}\right)$ $f'(x) > 0$ $\oplus$ve

$$f'(x) = \frac{1}{1 + (\sin x + \cos x)^2} \cdot (\cos x - \sin x) = \frac{\oplus}{\oplus} = \oplus\text{ve}.$$



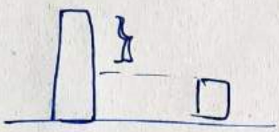
For $x \in \left(0, \frac{\pi}{4}\right)$ $\boxed{\cos x > \sin x}$

$f'(x) = \oplus$ve

Increasing

Example (51) Manufacturer can sell x items at a price of rupees $\left(5 - \frac{x}{100}\right)$ each. The cost price of x items in Rs $\left(\frac{x}{5} + 500\right)$. Find the number of items (x) he should sell to earn maximum profit.

Ans. Profit = S.P − ~~Cost~~ C.P.



S.P. = S(x)

$$= \underbrace{\left(5 - \frac{x}{100}\right)}_{\text{for 1 item}} \underbrace{x}_{x\text{-items}}$$

$$= \left(5x - \frac{x^2}{100}\right)$$

$$C.P. = C(x) = \left(\frac{x}{5} + 500\right)$$

$$\text{Profit} = P(x) = S(x) - C(x)$$

$$P(x) = 5x - \frac{x^2}{100} - \frac{x}{5} - 500$$

$$P(x) = \frac{24x}{5} - \frac{x^2}{100} - 500$$

maxima

$$P'(x) = \frac{24}{5} - \frac{x}{50} = 0 \quad (\text{For Critical Points})$$

$$\Rightarrow \frac{24}{\cancel{5}} = \frac{x}{\cancel{50}_{10}}$$

$$\Rightarrow \boxed{x = 240}$$

SODT

$$P''(x) = 0 - \frac{1}{50}$$

$$P''(240) = -\frac{1}{50} = \ominus ve$$

(local maxima)

∴ Manufacturer should sell 240 items to earn maximum profit.

Miscellaneous Exercise on Chapter ⑥

Q.1 Using differentials, find the approximate value of the following:

ⓐ $\left(\frac{17}{81}\right)^{\frac{1}{4}}$ ⓑ $(33)^{-\frac{1}{5}}$

Ans. ⓐ $\left(\frac{17}{81}\right)^{\frac{1}{4}}$

$81 = 3^4$

$16 = 2^4$

Function: $y = f(x) = x^{\frac{1}{4}}$

$$\boxed{y = \left(\frac{16}{81}\right)^{\frac{1}{4}} = \frac{2}{3}}$$

$$\left(\frac{17}{81}\right)^{\frac{1}{4}} = \left(\underbrace{\frac{16}{81}}_{x} + \underbrace{\frac{1}{81}}_{\Delta x}\right)^{\frac{1}{4}} = \underbrace{\frac{2}{3}}_{y} + \underbrace{\Delta y}_{?}$$

$$\boxed{\frac{\Delta y}{\Delta x} = \frac{dy}{dx}}$$

↓ Differtiation

$\because y = x^{\frac{1}{4}}$

$\frac{dy}{dx} = \frac{1}{4} \times x^{\frac{1}{4} - 1}$

$= \frac{1}{4} \cdot x^{-3/4}$

$\frac{dy}{dx} = \frac{1}{4 x^{3/4}}$

$\left(x = \frac{16}{81}\right)$

$\frac{dy}{dx} = \frac{1}{4\left(\frac{16}{81}\right)^{3/4}}$

$= \frac{1}{4\left(\frac{2}{3}\right)^{3}}$

$\Delta y = \frac{dy}{dx} \times \Delta x$

$\Delta y = \left(\frac{\cancel{27}}{4 \times 8}\right) \times \frac{1}{\cancel{81}_{3}} = \frac{1}{32 \times 3}$

$\Delta y = \frac{1}{96} = 0.0103$

$\left(\frac{17}{81}\right)^{\frac{1}{4}} = \frac{2}{3} + \Delta y$

$= 0.6666\ldots + 0.0103$

$= 0.667 + 0.010$

$= 0.677$ ✓

ⓑ $(33)^{-\frac{1}{5}}$

$y = f(x) = x^{-\frac{1}{5}}$

$$\underset{\downarrow}{(x}+\underset{\downarrow}{\Delta x)}^{-\frac{1}{5}} = y + \Delta y$$

$$(\underset{\underset{ⓧ}{\uparrow}}{32} + \underset{\underset{\Delta x}{\uparrow}}{1})^{-\frac{1}{5}} = ? = 0.5 + \Delta y$$

$$\frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

$$\Rightarrow \boxed{\Delta y = \frac{dy}{dx} \cdot \Delta x}$$

$y = x^{-\frac{1}{5}}$

$$\frac{dy}{dx} = -\frac{1}{5} \cdot x^{-\frac{1}{5}-1} = \frac{-1}{5} \cdot x^{-6/5}$$

$$= -\frac{1}{5 \cdot ⓧ^{6/5}}$$

$$\frac{dy}{dx} = -\frac{1}{5 \times (32)^{6/5}}$$

$$= -\frac{1}{5 \times (2^{\cancel{5}})^{6/\cancel{5}}}$$

$$= -\frac{1}{5 \times 64} = -\frac{1}{320}$$

$$\boxed{\begin{aligned} 2^5 &= 32 \\ 2 &= (32)^{\frac{1}{5}} \end{aligned}}$$

$x = 32, \ \Delta x = 1$

$y = x^{-\frac{1}{5}}$ $(x = 32)$

$y = (32)^{-\frac{1}{5}}$

$y = (2^5)^{-\frac{1}{5}} = 2^{-1} = \frac{1}{2} = 0.5$

$$\Delta y = \left(-\frac{1}{320}\right) \cdot (1)$$

$$= -\frac{1}{320}$$

≈ -0.003

$(33)^{-\frac{1}{5}}$

$= y + \Delta y$

$= 0.5 - 0.003$

$$\begin{array}{r} = 0.500 \\ -0.003 \\ \hline = 0.497 \end{array}$$

$\frac{1}{2} = 0.5$

$\frac{1}{4} = 0.25$

$\frac{1}{8} = 0.125$

$\frac{1}{16} = 0.0625$

$\frac{1}{32} = 0.03125$

$\frac{1}{320} = \underbrace{0.003}125$

$\hookrightarrow = 0.003$

[redacted] Class 12 maths. $\boxed{\text{AOD}}$

Miscellaneous Exercise on Chapter $\boxed{6}$

$\boxed{\text{Q.2}}$ Show that the function given by $f(x) = \frac{\log x}{x}$ has maximum at $x = e$.

Solution, $f(x) = \frac{\log x}{x}$ $\quad \left(\frac{u}{v}\right)'$

Domain = $\log x$ $\quad x > 0$

$$f'(x) = \frac{\left(\frac{1}{x}\right)\cdot x - (\log x)\cdot 1}{x^2}$$

$$f'(x) = \frac{1 - \log x}{x^2} = 0$$ (For Critical Points.) (x)

$\Rightarrow 1 - \log x = 0$

$\Rightarrow 1 = \log x$

$\Rightarrow 1 = \log_e x$ EXP

$\Rightarrow (e)^{(1)} = x$

$\Rightarrow \boxed{x = e}$ Critical point

Maths.

$\log a = \log_e a)$

$\log. \leftrightarrow \text{Expo.}$

$\log_a x \quad a^x$

$\log_a b = 100$

$b = (a)^{100}$

SODT

$$f'(x) = \frac{(1 - \log x)}{x^2}$$

$$f''(x) = \frac{\left(-\frac{1}{x}\right)\cdot x^2 - (1 - \log x)\cdot 2x}{(x^2)^2}$$

Put $x = e$

$\log_e e = 1$

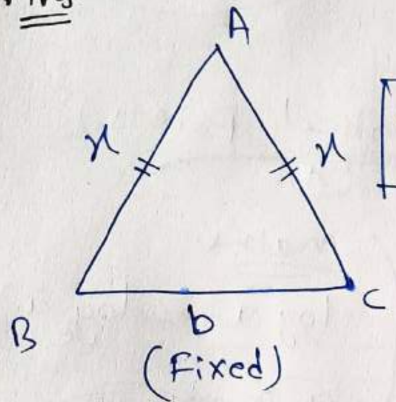
$$f''(e) = \frac{-e - 0}{e^4} = -\frac{1}{e^3} = \ominus ve$$ local maxima

$x = e$ Point of maxima

Miscellaneous Exercise on chapter ⑥

Q.3 Two equal sides of an isosceles triangle with fixed base b are decreasing at the rate of 3 cm/s. How fast is the area decreasing when the two equal sides are equal to the base?

Ans.

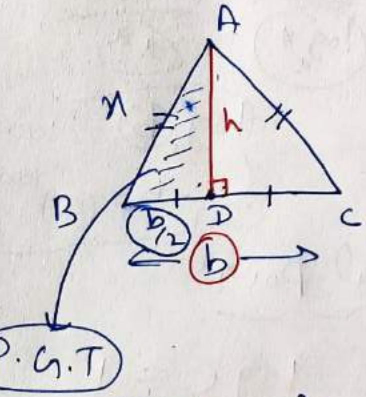


$\boxed{\frac{dx}{dt} = -3}$

$\frac{dA}{dt} = ?$ → when $\underbrace{x = x}= b$

Rate of change

✓D → Diagram
✓D → Derivative
✓F → Formula
✓D → Differentiation
C → Constant



$$x^2 = \left(\frac{b}{2}\right)^2 + h^2$$

$$\Rightarrow h = \sqrt{x^2 - \frac{b^2}{4}}$$

$$\text{Area} = \frac{1}{2}(\text{Base}) \times (\text{height})$$

$$A = \frac{1}{2}(b) \times (h)$$

$$A = \frac{1}{2} b \sqrt{x^2 - \frac{b^2}{4}}$$

by. diff. w.r.to 't'

$$\frac{dA}{dt} = \frac{b}{2} \cdot \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2 - \frac{b^2}{4}}} \cdot \overset{-3}{\frac{dx}{dt}}$$

when $x = b$

$$\frac{dA}{dt} = \frac{b}{2} \times \frac{b}{\sqrt{b^2 - \frac{b^2}{4}}} \cdot (-3)$$

$$\boxed{\frac{dA}{dt} = -\sqrt{3}\, b}$$

Area decreases @ $\sqrt{3}b$ cm²/s

Miscellaneous Exercise on Chapter ⑥

Q.4 Find the equation of the normal to curve $x^2 = 4y$ which passes through the point (1,2).

Ans.

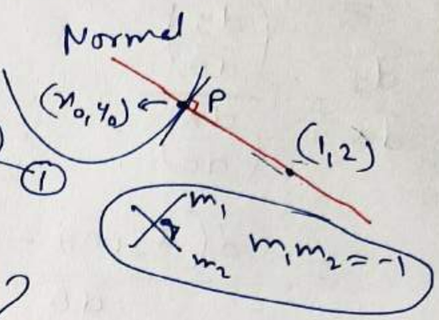
$x^2 = 4y$

⇒ $\cancel{2}x = \cancel{4}_{2} \frac{dy}{dx}$

⇒ $\boxed{\frac{dy}{dx} = \frac{x}{2}}$ = Slope of tangent

$x^2 = 4y$

$x_0^2 = 4y_0$ — ①



Slope of normal?

$m_N \cdot m_T = -1$

$m_N = -\frac{1}{m_T} = -\frac{2}{x_0}$

Slope of tangent at (x_0, y_0) $= \frac{x_0}{2} = m_T$

Equation of normal $(y - y_0) = m_N (x - x_0)$

⇒ $\boxed{y - y_0 = -\frac{2}{x_0}(x - x_0)}$

which passes through (1,2)   (put $x = 1$, $y = 2$)

⇒ $2 - y_0 = -\frac{2}{x_0}(1 - x_0)$

⇒ $\cancel{2x_0} - y_0 x_0 = -2 + \cancel{2x_0}$

⇒ $\boxed{x_0 y_0 = 2}$ — ② ⟶ $y_0 = \frac{2}{x_0}$

$\boxed{x_0^2 = 4y_0}$ — ①

by eqⁿ ① & ② —

⇒ $x_0^2 = 4\left(\frac{2}{x_0}\right)$

⇒ $x_0^3 = 8 = 2^3$

$\boxed{x_0 = 2}$

$x_0^2 = 4y_0$

⇒ $\cancel{2^2}_{1} = \cancel{4}\, y_0$

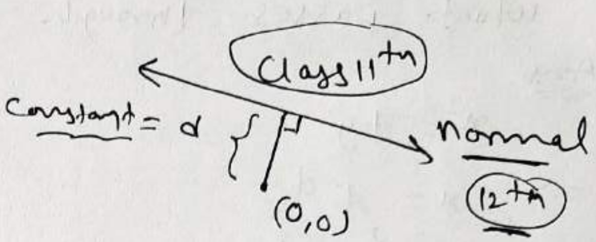
$\boxed{y_0 = 1}$ Point (2,1)

Normal: $(y - 1) = -\frac{2}{2}(x - 2)$

⇒ $\boxed{y + x - 3 = 0}$

Q.5 Show that the normal at any point 'θ' to the curve $x = a\cos\theta + a\theta\sin\theta$, $y = a\sin\theta - a\theta\cos\theta$, is at a constant distance from origin.

Ans.



$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}$$

$$= \frac{\frac{d(a\sin\theta - a\theta\cos\theta)}{d\theta}}{\frac{d(a\cos\theta + a\theta\sin\theta)}{d\theta}} = \frac{\cancel{a\cos\theta} - \cancel{a\cos\theta} + a\theta\sin\theta}{-\cancel{a\sin\theta} + \cancel{a\sin\theta} + a\theta\cos\theta}$$

$$\frac{dy}{dx} = \frac{\sin\theta}{\cos\theta} \checkmark$$

Slope of tangent (m_T)

Slope of normal $= m_N = -\frac{1}{m_T}$

$$\boxed{m_N = -\frac{\cos\theta}{\sin\theta}}$$

Point $(\underbrace{a\cos\theta + a\theta\sin\theta}_{x_0}, \underbrace{a\sin\theta - a\theta\cos\theta}_{y_0})$

Equation of normal.

$$(y - a\sin\theta + a\theta\cos\theta) = -\frac{\cos\theta}{\sin\theta}(x - a\cos\theta - a\theta\sin\theta)$$

$$\Rightarrow y\sin\theta - a\sin^2\theta + \cancel{a\theta\cos\theta\sin\theta} = -x\cos\theta + a\cos^2\theta + \cancel{a\theta\sin\theta . \cos\theta}$$

$$\Rightarrow x\cos\theta + y\sin\theta - a(\sin^2\theta + \cos^2\theta) = 0$$

$$\Rightarrow \boxed{x\cos\theta + y\sin\theta - a = 0}$$ Normal.

Normal   origin (0,0)

$x\cos\theta + y\sin\theta - a = 0$

$$d = \left|\frac{0+0-a}{\sqrt{\cos^2\theta+\sin^2\theta}}\right| = \left|\frac{-a}{\sqrt{1}}\right|$$

$$\underset{\uparrow}{d} = |-a| = \underset{\uparrow \text{Constant.}}{a}$$

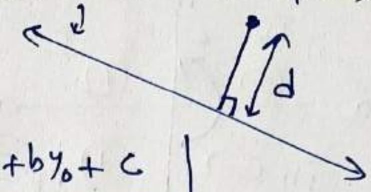
Distance b/w normal and origing

Distance Formula

b/w line & point

↓ ↓

$ax+by+c=0$ (x_0, y_0)



$$d = \left|\frac{ax_0+by_0+c}{\sqrt{a^2+b^2}}\right|$$

$\boxed{Q.6}$ Find the intervals in which the function

$$f(x) = \frac{4\sin x - 2x - x\cos x}{(2+\cos x)}$$ is (i) increasing. $f'(x)$ ⊕

(ii) decreasing. $f'(x)$ ⊖

Ans. $$f(x) = \frac{4\sin x - x(2+\cos x)}{(2+\cos x)}$$

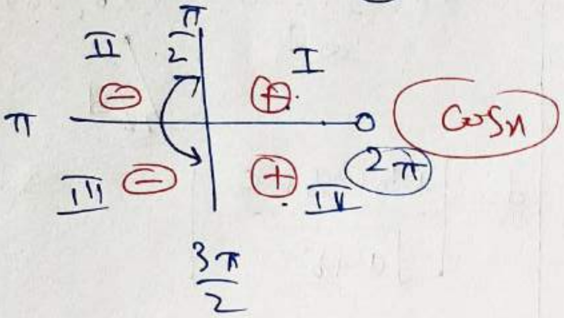
$$f(x) = \frac{4\sin x}{2+\cos x} - x$$

$$f'(x) = \frac{(4\cos x)(2+\cos x) - 4\sin x(-\sin x)}{(2+\cos x)^2} - 1$$

$$f'(x) = \frac{8\cos x + \cancel{4\cos^2 x + 4\sin^2 x} - \cancel{4} - \cos^2 x - 4\cos x}{(2+\cos x)^2}$$

$$f'(x) = \frac{4\cos x - \cos^2 x}{(2+\cos x)^2} = \frac{\cos x(4-\cos x)}{(2+\cos x)^2}$$

$$f'(x) = \frac{\cos x \overset{\oplus}{(4-\cos x)}}{\underset{\oplus}{(2+\cos x)^2}}$$



Factors

$(2+\cos x)^2 = \oplus ve$ Always

$\underset{[-1,1]}{(4-\cos x)} = \oplus ve$

Cos x → ⊖
↓
⊕

$x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \longrightarrow \cos x = \ominus ve \longrightarrow f'(x) = \ominus ve$ (Decreasing)

$x \in \left(0, \frac{\pi}{2}\right)$ or $x \in \left(\frac{3\pi}{2}, 2\pi\right) \rightarrow \cos x = \oplus ve \rightarrow f'(x) = \oplus ve$ Increasing.

Increasing	Decreasing
$x \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$	$x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

Q.7 Find the interval in which the function $f(x) = x^3 + \frac{1}{x^3}$, $(x \neq 0)$ is (I) increasing. $f'(x) = \oplus$
(II) Decreasing $f'(x) = \ominus$

Ans.

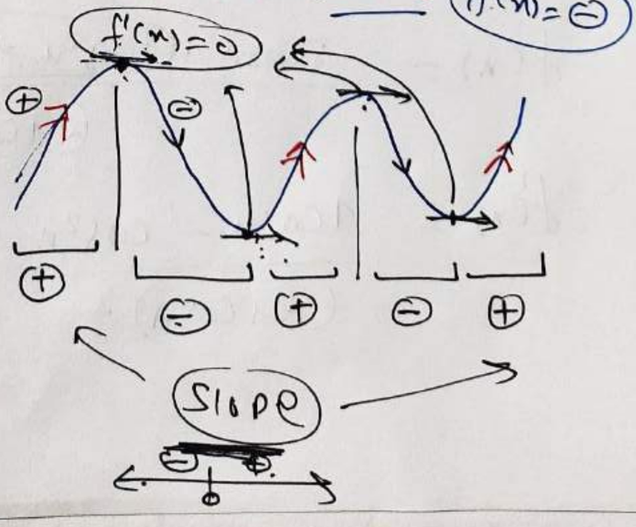
$f'(x) = 3x^2 - \frac{3}{x^4}$

$f'(x) = 3\left(\frac{x^6 - 1}{x^4}\right) = 0$ C.P.

$x^6 - 1 = 0$

$x^6 = 1$

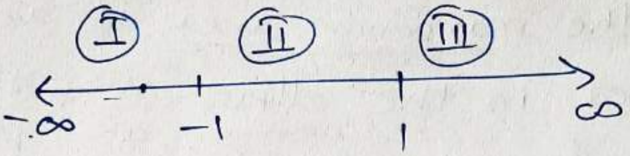
$\boxed{x = \pm 1}$



$$f'(x) = \frac{3(x^6-1)}{x^4}$$

Critical points

$\boxed{x = \pm 1}$ where $f'(x) = 0$



Interval	$f'(x)$ का sign	↑/↓
(I) $(-\infty, -1)$	$\oplus$	↑ (Increasing)
(II) $(-1, 1)$	$\ominus$	↓ (Dec.)
(III) $(1, \infty)$	$\oplus$	↑ (Increasing)

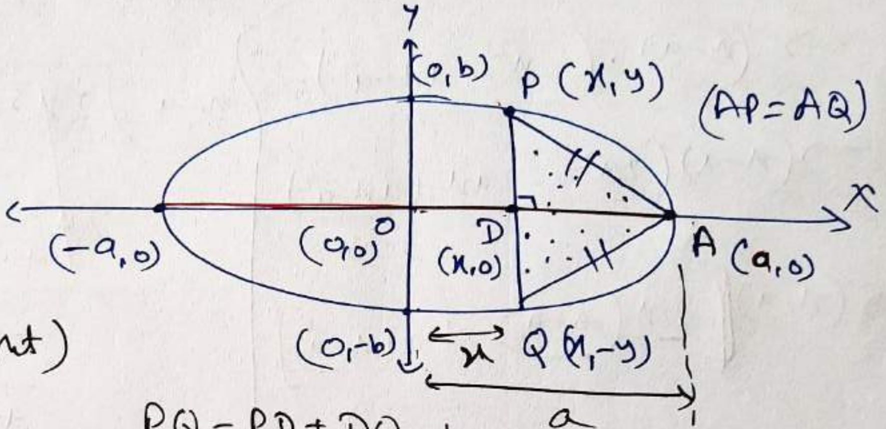
Increasing	Decreasing
$x \in (-\infty, -1) \cup (1, \infty)$	$x \in (-1, 1)$
$x < -1$ or $x > 1$	$-1 < x < 1$

Class 12 Maths (AOD)

Miscellaneous Exercise on Chapter ⑥

Q.8 Find the maximum area of an isosceles triangle inscribed in the ellipse $\boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}$ with its vertex at one end of the major axis.

Ans.



Area(ΔAPD)

$= A = \frac{1}{2}(\text{Base})(\text{Height})$

$A = \frac{1}{2}(PQ)(DA)$

$PQ = PD + DQ$
$= |y| + |y|$
$= 2y$

$AD = (a-x)$

$A = \frac{1}{\cancel{2}}\left|(\cancel{2}y)(a-x)\right|$

$\textcircled{A}\uparrow \quad \textcircled{A^2}\uparrow$

$\boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1} \Rightarrow \boxed{y^2 = b^2\left(1-\frac{x^2}{a^2}\right)}$

$$A^2 = \underline{y^2}(a-x)^2 = \underbrace{b^2\left(\frac{a^2-x^2}{a^2}\right)}\cdot(a^2+x^2-2ax)$$

$$\boxed{A^2 = \frac{b^2}{a^2}\left(a^4 + \cancel{a^2x^2} - 2a^3x - \cancel{a^2x^2} - x^4 + 2ax^3\right)}$$

$$\frac{d(A^2)}{dx} = \frac{b^2}{a^2}\left(-2a^3 \underline{- 4x^3} + 6ax^2\right)$$

$$\frac{d(A^2)}{dx} = -2\frac{b^2}{a^2}\left(2x^3 - 3ax^2 + a^3\right) = 0 \quad \left(\text{For Critical points}\right)$$

$\cancel{\frac{dy}{dx}} \Rightarrow \frac{d(A^2)}{dx} = -\frac{2b^2}{a^2}\left(2x^3-3ax^2+a^3\right)=0$

$2x^3-3ax^2+a^3=0$

$\boxed{x=a}$ satisfy

$(x-a)$ factor

$\Rightarrow (x-a)\cdot\left(2x^2-ax-a^2\right)=0$

$\Rightarrow (x-a)\left(2x^2-2ax+ax-a^2\right)=0$

$\Rightarrow (x-a)\left[2x(x-a)+a(x-a)\right]=0$

$\Rightarrow (x-a)^2(2x+a)=0$

$\cancel{x=a}$, $-\frac{a}{2}$ (Critical Points)

$x-a\ \big)\ 2x^3-3ax^2+a^3\ \big(\ 2x^2-ax-a^2$

$\underset{-}{2x^3}\ \underset{+}{-2ax^2}$

$-ax^2$

$\underset{+}{-ax^2}\ \underset{-}{+a^2x}$

$-a^2x+a^3$

$\underset{+}{-a^2x}\ \underset{-}{+a^3}$

0

SODT

$\frac{d(A^2)}{dx} = -\frac{2b^2}{a^2}\left(2x^3-3ax^2+a^3\right)$

$\frac{d^2(A^2)}{dx^2} = -\frac{2b^2}{a^2}\left(6x^2-6ax\right)$

Put $\left(x=-\frac{a}{2}\right)$

$\frac{d^2(A^2)}{dx^2} = \ominus\left(\oplus+\oplus\right)$

$= \ominus$ve

Maxima

$x=-\frac{a}{2}$ maxima

$A^2=\frac{b^2}{a^2}\left(a^4-2a^3x-x^4+2ax^3\right)$

$A^2=\frac{b^2}{a^2}\left(a^4+a^4-\frac{a^4}{16}-\frac{a^4}{4}\right)$

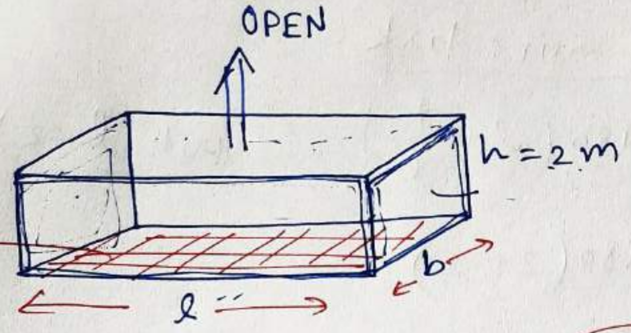
$A^2=\frac{b^2}{a^2}\left(\frac{32-1-4}{16}\right)a^4$

$A^2=\frac{27}{16}a^2b^2$ ↑

$\boxed{A=\frac{3\sqrt{3}}{4}ab}$ ↑

Q.9

cuboid



l×b

Volume = 8 m3

$lbh = 8$

$lb(2) = 8$

$lb = 4$ = Area (Base)

Total cost = Area × Rate

Rate of cost for base = ₹ $70/m^2$ → Total cost of Base = 4×70

Rate of cost for sides (walls) = ₹ $45/m^2$ → Total cost of Sides = $4(l+b).45$

Area of sides $= 2(lh)+2(bh) = 2h(l+b)$
$= 4(l+b)$

least

Total ~~Exped~~ Expenditure on tank = E = 280 + 180(l+b)

$E = 280 + 180(l+b)$

$$\boxed{E = 280+180\left(l+\frac{4}{l}\right)} \quad \left(\because lb=4,\ b=\frac{4}{l}\right)$$

$$\frac{d(E)}{dl} = 180\left(1-\frac{4}{l^2}\right) = 0 \text{ (For Critical points)}$$

$$1 = \frac{4}{l^2}$$

$$l^2 = 4 \Rightarrow l = \pm 2 \quad l=-2 \times, \quad l=2$$

SODT

$$\frac{d^2(E)}{dl^2} = 180\left(\frac{8}{l^3}\right)$$

If $l=2$, $\frac{d^2(E)}{dl^2} = \oplus$

$\boxed{l=2}$ Point of minima

Cost of least expensive tank

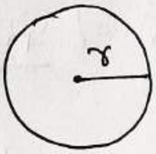
$E = 280 + 180\left(l + \frac{4}{l}\right)$ (put $\underline{l=2}$)

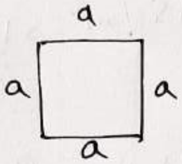
$E = 280 + 180(2+2)$

$E = 280 + 720 = \underline{1000}$ ₹

* * *

$\boxed{\text{Q.10}}$





To Prove $\boxed{a = 2r}$

$2\pi r + 4a = K$ (Constant)

$\boxed{a = \frac{K - 2\pi r}{4}}$

$A = \pi r^2 + a^2$

$A = \pi r^2 + \left(\frac{K-2\pi r}{4}\right)^2 = \pi r^2 + \frac{K^2 + 4\pi^2 r^2 - 4\pi K r}{16}$

$A = \frac{16\pi r^2 + K^2 + 4\pi^2 r^2 - \cancel{4K}\, 4\pi K r}{16}$

least

$\frac{dA}{dr} = \frac{32\pi r + 8\pi^2 r - 4\pi K}{16} = 0$ For C.P.

$4\pi(8r + 2\pi r - K) = 0$

$\Rightarrow \boxed{r = \frac{K}{8 + 2\pi}}$

SODT

$\frac{d^2A}{dr^2} = \frac{32\pi + 8\pi^2}{16}$

Area

$= (+)$ minima

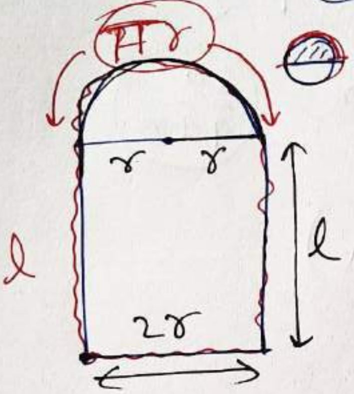
we have

$$\underbrace{2\pi r + 4a = \boxed{K}} \qquad \underbrace{r = \frac{K}{8+2\pi}}$$

$$r = \frac{2\pi r + 4a}{8+2\pi} \Rightarrow 8r + 2\cancel{\pi} r = 2\cancel{\pi} r + 4a$$

$$\Rightarrow \boxed{2r = a} \checkmark$$

$\boxed{\text{Q.11}}$



total Perimeter = 10 m.

$$2l + 2r + \pi r = 10$$

$$l = \frac{10 - 2r - \pi r}{2} \checkmark$$

$$\boxed{\text{Maximum light passing} = \text{maximum area}}$$

A = Area of reectangle + Area of Circle

$$A = 2rl + \frac{\pi r^2}{2}$$

$$A = \cancel{2}r\left(\frac{10 - 2r - \pi r}{\cancel{2}}\right) + \frac{\pi r^2}{2}$$

$$A = 10r - 2r^2 - \pi r^2 + \frac{\pi r^2}{2}$$

$$A = 10r - 2r^2 - \frac{\pi r^2}{2}$$

$$\frac{dA}{dr} = 10 - 4r - \pi r = 0 \quad \boxed{\text{For C.P.}}$$

$$10 = r(4+\pi)$$

$$\boxed{r = \frac{10}{4+\pi}}$$

$$\frac{dA}{dr} = 10 - 4r - \pi r$$

$\boxed{C.P.}$

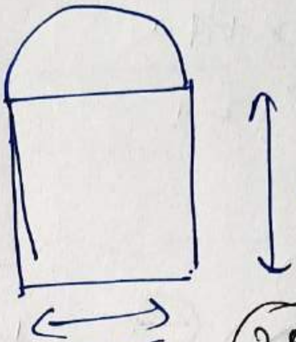
$$r = \frac{10}{4+\pi}$$

$$\frac{d^2A}{dr^2} = -4 - \pi$$

$$= \boxed{-}ve$$

maxima

Point of maxima



$$l = \frac{10}{4+\pi}$$

$$2r = \boxed{\frac{20}{4+\pi}}$$

$$l = \frac{10 - 2r - \pi r}{2} \leftarrow \left(\text{Put } r = \frac{10}{4+\pi}\right)$$

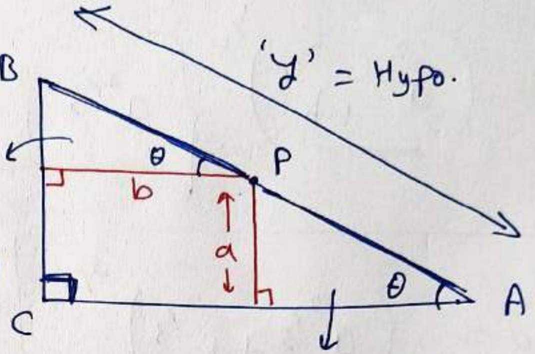
$$l = \frac{10 - \frac{20}{4+\pi} - \frac{10\pi}{4+\pi}}{2} = \frac{40 + \cancel{10\pi} - 20 - \cancel{10\pi}}{2(4+\pi)}$$

$$\boxed{l = \frac{10}{4+\pi}}$$

Miscellaneous Exercise on Chapter-6

Q.12 A point on the hypotenuse of a triangle is at a distance a and b from the sides of the triangle. Show that the minimum length of hypotenuse is $(a^{2/3}+b^{2/3})^{3/2}$.

Ans.



$\cos\theta = \frac{b}{BP}$

$BP = b\sec\theta$

$\sin\theta = \frac{a}{AP}$.

$AP = a\,\text{cosec}\,\theta$

$y = AP + PB$

$y = a\,\text{cosec}\,\theta + b\sec\theta$

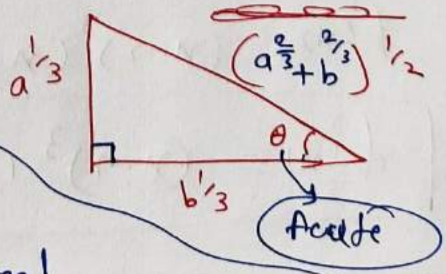
$\frac{dy}{d\theta} = -a\,\text{cosec}\,\theta\cdot\cot\theta + b\sec\theta\cdot\tan\theta = 0$ (For Critical Points)

$b\cdot\frac{1}{\cos\theta}\cdot\frac{\sin\theta}{\cos\theta} = a\cdot\frac{1}{\sin\theta}\cdot\frac{\cos\theta}{\sin\theta}$

$\Rightarrow b\cdot\sin^3\theta = a\cos^3\theta$

$\Rightarrow \tan^3\theta = \frac{a}{b}$

$\tan\theta = \frac{a^{1/3}}{b^{1/3}} = \frac{\text{Perp.}}{\text{Base.}}$

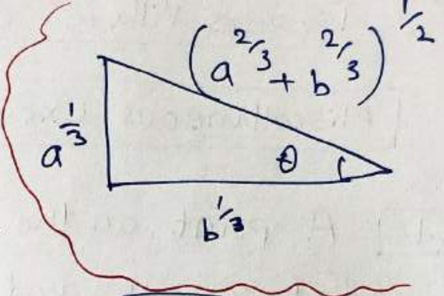


SODT

$\frac{d^2y}{d\theta^2} = +a\,\text{cosec}\,\theta\cdot\cot^2\theta + a\,\text{cosec}^3\theta + b\sec\theta\cdot\tan^2\theta + b\sec^3\theta$

If we put the value of critical Point we get $\frac{d^2y}{d\theta^2} = +ve$ minima

Hypotenuse $y = a\,\mathrm{cosec}\theta + b\sec\theta$



minimum

$$y = a\frac{(a^{2/3}+b^{2/3})^{1/2}}{a^{1/3}} + b.\frac{(a^{2/3}+b^{2/3})^{1/2}}{b^{1/3}}$$

$\mathrm{Cosec}\,\theta = \frac{H}{P}$

$\sec\theta = \frac{H}{B}$

$$y = \left(a^{2/3}+b^{2/3}\right)^{1/2}.\left\{a^{2/3}+b^{2/3}\right\}^{1}$$

$$\boxed{y = \left(a^{2/3}+b^{2/3}\right)^{3/2}}$$

↑ Hyp. (minimum).

Q.13 Find the points at which the function $f(x) = (x-2)^4.(x+1)^3$ has (i) local minima (ii) local maxima (iii) point of inflection.

FODT ✓
SODT ✗

Ans. $f(x) = (x-2)^4 (x+1)^3$

$f'(x) = 4.(x-2)^3 (x+1)^3 + 3(x-2)^4.(x+1)^2$

$f'(x) = (x-2)^3 (x+1)^2.[4x+4+3x-6]$

$\boxed{f'(x) = (x-2)^3 (x+1)^2 (7x-2)} = 0$ (For Critical Points)

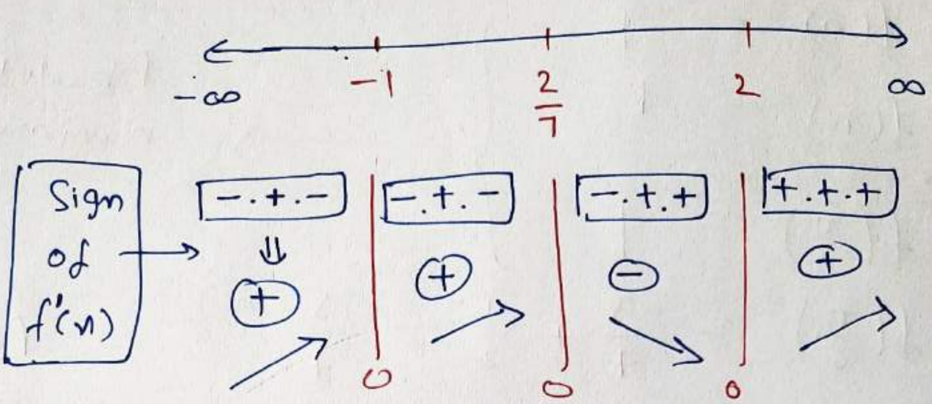
$x = 2,\ x = -1,\ x = \frac{2}{7}$

$f'(x)=(x-2)^3(x+1)^2(7x-2)=0 \longrightarrow \boxed{-1, \frac{2}{7}, 2} \longrightarrow \underline{f'(x)=0}$

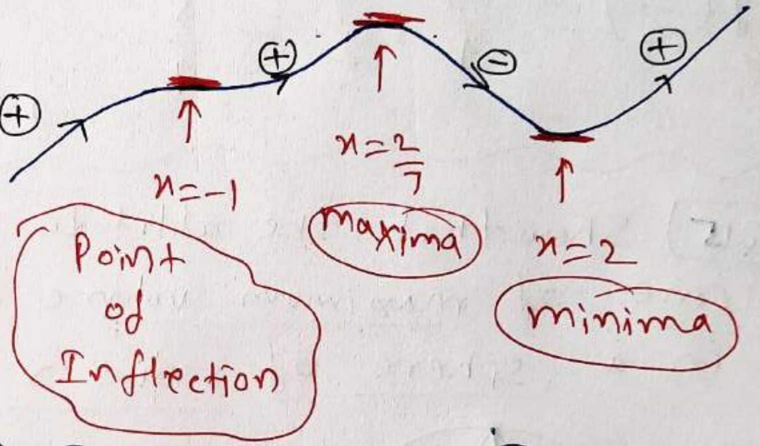
Slope

FoDT

↗ ↘



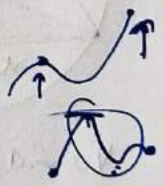
Graph →



Q.14 Find the absolute maximum and minimum values of $f(x)=\cos^2 x+\sin x,\ x\in[0,\pi]$.

Ans.

$f'(x)=2\cos x\cdot(-\sin x)+\cos x=0$

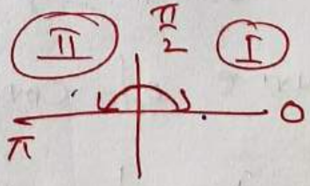


For C.P.

$\cos x(-2\sin x+1)=0$

$\cos x=0$, $\sin x=\frac{1}{2}$

$x=\frac{\pi}{2}$, $x=\frac{\pi}{6}$ $x=\frac{5\pi}{6}$



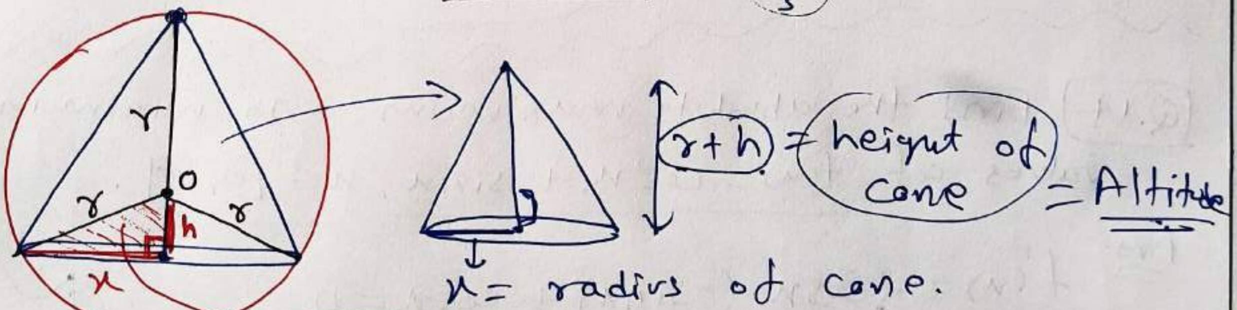
$\boxed{f(x) = \cos^2 x + \sin x}$, $x \in [0, \pi]$

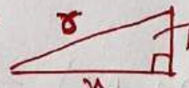
$$\left.\begin{aligned} f(0) &= 1 \\ f(\pi) &= 1 \\ f\left(\tfrac{\pi}{2}\right) &= 1 \end{aligned}\right\} \longrightarrow \left(\text{Absolute minimum} = 1 \right)$$

$$\left.\begin{aligned} f\left(\tfrac{\pi}{6}\right) &= \tfrac{3}{4} + \tfrac{1}{2} = \tfrac{5}{4} \\ f\left(\tfrac{5\pi}{6}\right) &= \tfrac{3}{4} + \tfrac{1}{2} = \tfrac{5}{4} \end{aligned}\right\} \longrightarrow \text{Absolute maximum} \;\; \| \;\; \tfrac{5}{4}$$

Q.15 Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$

Ans.





→ P.G.T. $x^2 = r^2 - h^2$

Variable = x, h

Constant = r

volume of cone = $V = \frac{1}{3}\pi x^2 \cdot (r+h)$

$$V = \frac{1}{3}\pi (r^2 - h^2)(r+h)$$

$$\boxed{V = \frac{1}{3}\pi (r^3 + r^2 h - r h^2 - h^3)}$$

Maxima → Diff.

$v = \frac{1}{3}\pi\left(r^2 + r^2 h - r h^2 - h^3\right)$

$\downarrow$ variable → (h), v

Constant → r

$\frac{dv}{dh} = \frac{1}{3}\pi\left(r^2 - 2rh - 3h^2\right) = 0$ For C.P.

$= -\frac{1}{3}\pi\left(3h^2 + \underline{2rh} - r^2\right) = 0$

$= -\frac{1}{3}\pi\left(\underbrace{3h^2 + 3rh}_{} \underbrace{- rh - r^2}_{}\right) = 0$

③ SODT

$\Rightarrow 3h(h+r) - r(h+r) = 0$

$(h+r).(3h-r) = 0$

$\cancel{h = -r}$ $\quad$ $h = \frac{r}{3}$ C.P.

$\frac{d^2v}{dh^2} = \frac{1}{3}\pi\left(-2r - 6h\right)$

Put $h = \frac{r}{3}$ → P $h = \frac{r}{3}$

Point of maxima

$\frac{d^2v}{dh^2} = \frac{1}{3}\pi\left(-2r - 6.\frac{r}{3}\right)$

$= \ominus$ve

maxima

Altitude of cone (Height) $= r + h$

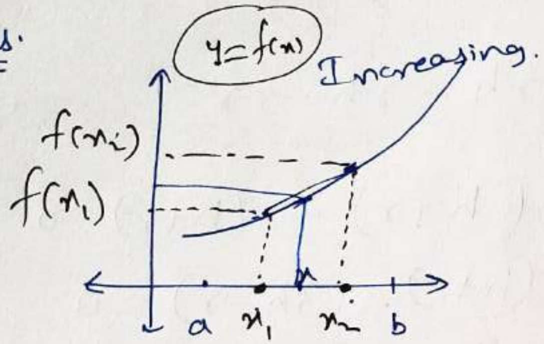
$= r + \frac{r}{3}$

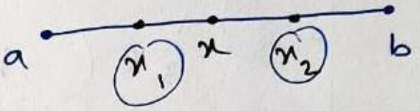
$= \frac{4r}{3}$

Miscellaneous Exercise on Chapter 6

Q.16 Let f be a function defined on $[a,b]$ such that $f'(x) > 0$, for all $x \in (a,b)$. Then prove that f is an increasing function on (a,b).

Ans.





let $x_1, x_2 \in [a,b]$ $x_1 < x_2$

let $x \in [x_1, x_2]$

To Prove: If $x_1 < x_2$

then $f(x_1) < f(x_2)$

Inc.

Mean value Theorem

Proof: $f'(x) = \dfrac{f(x_2)-f(x_1)}{x_2-x_1}$

$\because f'(x) > 0$

$\Rightarrow \dfrac{f(x_2)-f(x_1)}{x_2-x_1} > 0$

↑ ↓

$\Rightarrow \boxed{f(x_2) - f(x_1) > 0}$

$\boxed{f(x_2) > f(x_1)}$

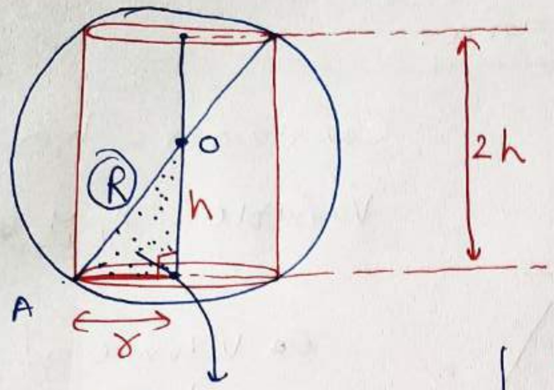
$\therefore$ Here if $x_1 < x_2$ then

$f(x_1) < f(x_2)$

$\therefore f$ is increasing function.

Q.17 Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find its maximum volume.

Ans.



P.G.T.

$R^2 = r^2 + h^2$

$\Rightarrow \boxed{r^2 = R^2 - h^2}$

To Prove: $2h = \frac{2R}{\sqrt{3}}$

Volume of Cylinder

$V = \pi r^2 (2h)$

$\boxed{V = \pi(R^2 - h^2).2h}$

$V = 2\pi(R^2h - h^3)$

maxima → Diff.

For C.P.

$\frac{dv}{dh} = 2\pi(R^2 - 3h^2) = 0$

$R^2 = 3h^2$

$\sqrt{\frac{R^2}{3}} = h$

$\boxed{h = \frac{R}{\sqrt{3}}}$

SODT

$\frac{d^2v}{dh^2} = 2\pi(-6h) = \ominus ve$ maxima

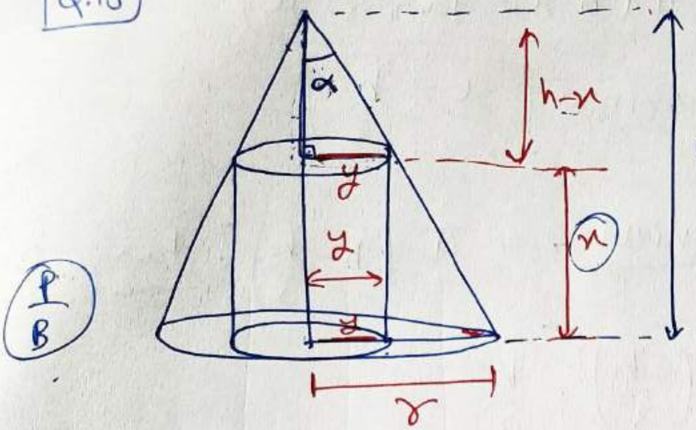
$h = \frac{R}{\sqrt{3}}$

Height of cylinder $= 2h = \frac{2R}{\sqrt{3}}$

Volume of cylinder $= V = \pi(R^2 - h^2)2h$

$V = \pi\underbrace{\left(R^2 - \frac{R^2}{3}\right)}\,2 \times \frac{R}{\sqrt{3}} = \frac{4\pi R^3}{3\sqrt{3}}$

Q.18



$\tan\alpha = \frac{y}{h-x} = \frac{r}{h}$

Small Δ

$\Rightarrow \boxed{y = (h-x).\tan\alpha}$

$\boxed{r = h.\tan\alpha}$

To Prove

Height of cylinder = $\frac{1}{3}$(Height of Cone)

$\Downarrow$

$\boxed{x = \frac{h}{3}}$ —

$\boxed{V = \frac{4}{27}\pi h^3 \tan^2\alpha}$

maximum

Diff.

Volume of Cylinder

$V = \pi y^2 . x$

$\boxed{V = \pi (h-x)^2 \tan^2\alpha . x}$

$V = \pi \tan^2\alpha \, (h^2x - 2hx^2 + x^3)$

Diff.

$\frac{dv}{dx} = \pi.\tan^2\alpha \, (h^2 - 4hx + 3x^2)$

(For Critical Points)

$\frac{dv}{dx} = 0$

$\Rightarrow h^2 - 4hx + 3x^2 = 0$

$\Rightarrow \underbrace{h^2 - 3hx} - \underbrace{hx + 3x^2} = 0$

$\Rightarrow h(h-3x) - x(h-3x) = 0$

$\Rightarrow (h-x)\,(h-3x) = 0$

$\cancel{h = x}$ $\quad$ $x = h/3$ C.P.

SODT

$\frac{d^2v}{dx^2} = \pi \tan^2\alpha \, .(-4h + 6x)$

Put $x = \frac{h}{3}$ → Proved

$\frac{d^2v}{dx^2} = \underbrace{\pi \tan^2\alpha \, (-4h + 2h)}$

(−)ve

maxima

volume of cylinder $\boxed{V = \pi (h-x)^2 \tan^2\alpha \cdot x}$

Point of maxima $\boxed{x = \frac{h}{3}}$

maximum volume of cylinder $= \pi \left(h - \frac{h}{3}\right)^2 \cdot \tan^2\alpha \cdot \frac{h}{3}$

$$= \pi \tan^2\alpha \cdot \left(\frac{2h}{3}\right)^2 \cdot \frac{h}{3}$$

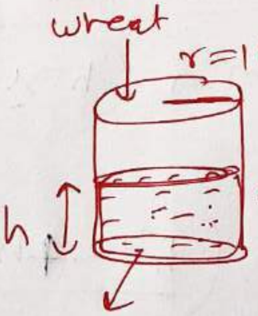
$$= \frac{4\pi}{27} \tan^2\alpha \cdot h^3$$

Q.19 A cylindrical tank of radius 10 m is being filled with wheat at the rate of 314 cubic meter per hour. Then the depth of the wheat is increasing at the rate of

[✓ (A) 1 m/h] (B) 0.1 m/h (C) 1.1 m/h (D) 0.5 m/h.

ROC

D →
D →
F —
D →
C



$314 \text{ m}^3/\text{h} = \frac{dV}{dt}$

$\frac{dh}{dt} = ?$

$V = \pi r^2 h$

$V = \pi(100) \cdot h$

$\frac{dV}{dt} = \pi(100) \frac{dh}{dt}$

$\Rightarrow 314 = (3.14)(100) \cdot \frac{dh}{dt}$

$\Rightarrow \frac{dh}{dt} = \frac{314}{314} = 1 \text{ m/h}$

Miscellaneous Exercise on Chapter ⑥

Q.20 The Slope of the tangent to the curve $x = t^2 + 3t - 8$, $y = 2t^2 - 2t - 5$ at the point $(2,-1)$ is -

(A) $\frac{22}{7}$ [✓ (B) $\frac{6}{7}$] (C) $\frac{7}{6}$ (D) $\frac{-6}{7}$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{4t-2}{2t+3} = \boxed{\text{Slope of tangent}} = \boxed{\frac{6}{7}}$$

$\boxed{t=2}$

Point $\rightarrow (2,-1)$

$\uparrow x \quad \uparrow y$

$t^2+3t-8 = \boxed{2}$	$2t^2-2t-5 = \boxed{-1}$
$\Rightarrow t^2+3t-10=0$	$2t^2-2t-4=0$
$\Rightarrow \underbrace{t^2+5t}-\underbrace{2t-10}=0$	$t^2-t-2=0$
$\Rightarrow t(t+5)-2(t+5)=0$	$\Rightarrow \underbrace{t^2+t}-\underbrace{2t-2}=0$
$(t-2)(t+5)=0$	$\Rightarrow t(t+1)-2(t+1)=0$
$t=\boxed{2},-5$	$(t+1)(t-2)=0$
	$t=-1,\boxed{2}$

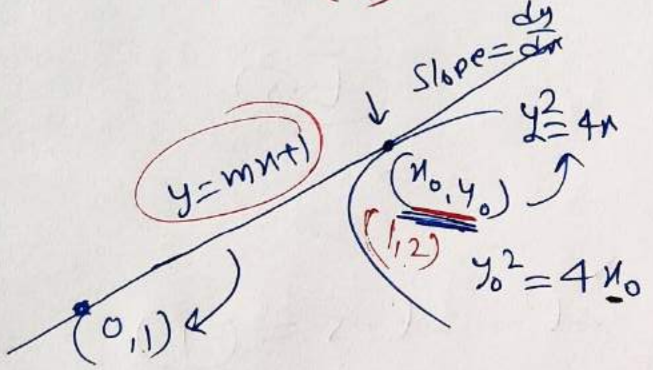
$\boxed{t=2}$

$\boxed{Q.21}$ The line $\boxed{y=mx+1}$ is a tangent to the curve $\boxed{y^2=4x}$ if the value of m is –

(A) 1 (B) 2 (C) 3 (D) $\frac{1}{2}$

I

$\boxed{y=mx+1}$ always passes through $(0,1)$



$$y^2=4x$$

$$\Rightarrow 2y.\frac{dy}{dx}=4$$

Slope of tangent at (x_0,y_0) $=\frac{dy}{dx}=\frac{2}{y}=\boxed{\frac{2}{y_0}}=m$

Tangent: $(y-y_0)=\frac{2}{y_0}(x-x_0)$

Passes through $(0,1)$ $\left(\because y_0^2=4x_0\right)$

$$\Rightarrow (1-y_0)=\frac{\cancel{2}}{\cancel{y_0}}\left(0-\frac{y_0^{\cancel{2}}}{\cancel{4}_{2}}\right)$$

$$\Rightarrow (1-y_0)=\frac{-y_0}{2} \Rightarrow$$

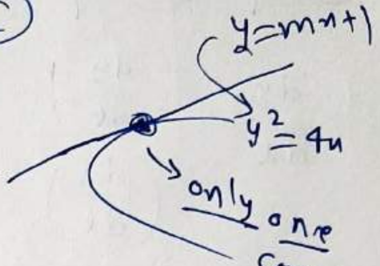
$$\Rightarrow 2-2y_0=-y_0$$

$$\Rightarrow \boxed{2=y_0} \quad (1,2)$$

$$4=4x_0$$

$$\boxed{x_0=1} \quad \boxed{m=\frac{2}{y_0}=\frac{2}{2}}=1$$

II



only one common point

$$y^2=4x$$

$$\Rightarrow (mx+1)^2=4x$$

$$\Rightarrow m^2x^2+2mx+1=4x$$

$$\Rightarrow \underline{m^2x^2+(2m-4)x+1=0}$$

Quadratic Δ Same roots

$\boxed{D=0}$

$$b^2-4ac=0$$

$$(2m-4)^2-4m^2.(1)=0$$

$\Rightarrow$ Solve

$$\boxed{m=1}$$

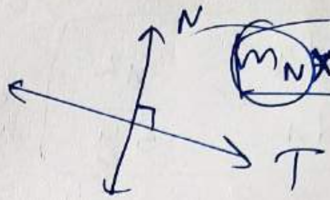
$\boxed{Q.22}$ The normal at the point (1,1) on the curve $2y + x^2 = 3$ is —

(A) $x+y=0$ [✓ (B)] $x-y=0$ (C) $x+y+1=0$ (D) $x-y=0$

Ans.

$$\frac{2dy}{dx} + 2x = 0$$

$$\Rightarrow \boxed{\frac{dy}{dx} = -x} \leftarrow (1,1)$$



$\boxed{m_N \times m_T = -1}$

Slope of tangent at $(1,1) = m_T = -1$

Slope of normal at $(1,1) = m_N = \frac{-1}{m_T}$

$$m_N = \frac{-1}{-1} = 1$$

Normal.

$$(y-1) = (1)(x-1) \Rightarrow y - \cancel{1} = x - \cancel{1}$$

$$\Rightarrow \boxed{x - y = 0}$$

$\boxed{Q.23}$ = $\boxed{Q.4}$ = $\boxed{\text{Example } \boxed{45}}$

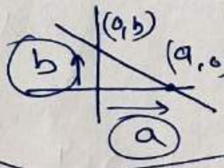
$\boxed{Q.24}$ The point on the curve $9y^2 = x^3$, where the normal to the curve makes equal intercepts with the axes are —

[✓ (A)] $\left(4, \pm\frac{8}{3}\right)$ (B) $\left(4, -\frac{8}{3}\right)$ (C) $\left(4, \pm\frac{3}{8}\right)$ (D) $\left(\pm 4, \frac{3}{8}\right)$

Intercept form

$$\frac{x}{a} + \frac{y}{b} = 1$$

$a = b$



equal intercept

$$\frac{x}{a} + \frac{y}{a} = 1$$

$$\Rightarrow \boxed{x + y = a}$$

$y = -x + a$

$m = -1$

Curve

$9y^2 = x^3$

$\Rightarrow \cancel{9}^{3} \times 2y \cdot \frac{dy}{dx} = \cancel{3}x^2$

$\Rightarrow \boxed{\frac{dy}{dx} = \frac{x^2}{6y}}$ slope of tangent

m_T

Slope of normal $= m_N = -\frac{1}{m_T} = -\frac{6y_0}{x_0^2}$

Equal intercept

$m = -1$

$\frac{x}{a} + \frac{y}{a} = 1$

normal makes equal intercepts with axes.

$m_N = \frac{-6y_0}{x_0^2} = -1$

$\Rightarrow 6y = x^2$ —①

$\boxed{y_0 = \frac{x_0^2}{6}}$

$9y^2 = x^3$

(x_0, y_0)

$9y_0^2 = x_0^3$ —②

$9\left(\frac{x_0^2}{6}\right)^2 = x_0^3$

$\Rightarrow \cancel{9}\frac{x_0^4}{\cancel{36}\,4} = x_0^3$

$\Rightarrow x_0^4 = 4x_0^3$

$\Rightarrow x_0^3(x_0 - 4) = 0$

$x_0 = 0$ $\quad$ $x_0 = 4$

$\boxed{y_0 = 0}$ ✗

$\boxed{x_0 = 4}$

$6y_0 = x_0^2$

$6y_0 = 16$

$y_0 = \frac{8}{3}$

$\left(4, \frac{8}{3}\right)$

$9y_0^2 = x_0^3$

$9y_0^2 = 64$

$y_0^2 = \frac{64}{9}$

$y_0 = \pm\frac{8}{3}$

$\left(4, \pm\frac{8}{3}\right)$

$\left(4, \frac{8}{3}\right) \rightarrow \left(y - \frac{8}{3}\right) = -(x-4)$ $\quad m=-1$

$\left(4, -\frac{8}{3}\right) \rightarrow (y + 8/3) = +(x-4)$ $\quad m=+1$